On the Strategic Advantages of a Lack of Common Knowledge

Barton L. Lipman
GSIA
Carnegie Mellon University
Pittsburgh, PA 15213

ABSTRACT

I illustrate the surprising strategic effects a lack of common knowledge can have in a simple example. I analyze a two-person bargaining game in which the seller makes a single take-it-or-leave-it offer. The seller knows everything except possibly what information the buyer has. If it is common knowledge that the buyer knows the value of the object or if it is common knowledge that the buyer does not know the value, then the buyer's expected payoff is zero. However, if the seller does not know what the buyer knows, then the buyer may be able to obtain some of the gains from trade. In the exogenous information case, where there is a fixed probability q that the buyer is informed, if q is far enough from zero and one, the buyer's expected payoff is strictly positive. However, if the seller is almost sure of what the buyer knows, the buyer's expected payoff is zero. Surprisingly, the endogenous information case, where, after seeing the seller's offer, the buyer decides whether or not to become informed, is quite different. If the cost of obtaining information is zero, we get the same outcome as it is common knowledge that the buyer is informed. However, if the cost of obtaining the information is small but strictly positive, the buyer's payoff is strictly positive. As the cost goes to zero, the limit of the buyer's payoff is strictly positive and may even be virtually all of the gains from trade.

2. INTRODUCTION

Much research in economics over the past decade has demonstrated the fact that the distribution of knowledge is of crucial importance for predicting outcomes. Changing the specification of "who knows what and when they learn it" can greatly affect the equilibrium outcome.[1] What is less well understood is how knowledge about the knowledge of others affects equilibrium outcomes. (The small amount of literature on this subject is discussed below.) Typically, we assume that "who knows what and when they learn it" is common knowledge. Informally, a statement p is said to be common knowledge among the players if all statements of the form "everyone knows that everyone knows that ... knows that p" are true. In this paper, I show by example that relaxing the assumption that "who knows what" is common knowledge can have very surprising implications.

[1] For classic examples, see Kreps and Wilson [1982a] and Milgrom and Roberts [1982].

I consider a simple two–player bargaining game. Throughout, the seller is taken to be informed about everything except possibly the information of the buyer. The seller makes a single take–it–or–leave–it offer, which the buyer can accept or reject. It is easy to see that when it is common knowledge that the buyer is perfectly informed, the seller's bargaining power enables him to extract all the gains from trade. It is also straightforward to show that when it is common knowledge that the buyer is imperfectly informed, the buyer, again, receives none of the gains from trade. I relax the assumption that the seller knows what the buyer knows in two ways.

First, I consider the case of exogenous information, Here the information of the buyer is exogenously determined—the buyer is either informed or not and the seller does not know which. I show that the buyer obtains some of the gains from trade if the seller is sufficiently unsure about what the buyer knows. However, near certainty, the outcome is continuous in the probability that the buyer is informed. If the seller is almost sure that the buyer is informed or almost sure that the buyer is uninformed, the buyer's share of the surplus is small or zero.

The second case is endogenous information. Here the buyer has the option of becoming informed at a cost. Surprisingly, the outcome is *not* continuous in the cost at zero. The outcome when the cost of becoming informed is zero is the same as the outcome when it is common knowledge that the buyer is informed. For any strictly positive cost, however, the buyer's expected payoff is bounded away from zero. In fact, in some cases, the buyer receives virtually all of the surplus when the cost of becoming informed is close to zero! As discussed in Lipman [1989], one can view a game with endogenous information acquisition as a model of bounded rationality. Under this interpretation, the example indicates that complete rationality and "almost complete rationality" can be very different in extensive form games.

In Section 3, I set out the model, briefly discuss the equilibrium when the buyer's information is common knowledge, and analyze the exogenous information case. Section 4 considers the endogenous information case and offers concluding remarks. Proofs are in Section 5.

A Comment on Interpretation. A number of authors have argued that common knowledge of the information structure is without loss of generality. Let S_0 be the set of all possible states of the world. Suppose we represent the information of player i about S_0 by a partition of S_0. Aumann [1987] gives the following argument that the assumption that the partitions are common knowledge is without loss of generality. Suppose that player j thinks that player i's partition could be either Π_1 or Π_2. Then it must be true that there is a state of the world in which i's partition is Π_1 and another in which it is Π_2 and player j cannot tell these states apart. That is, the collection of states in S_0 is not a complete specification of the set of states of the world. In particular, we require a bigger state space in which we can represent the uncertainty about the knowledge of others regarding S_0. In the bigger space, say S_1, the information structure regarding S_1 will be common knowledge if S_1 is appropriately chosen.[2] In this sense, a lack of common knowledge at one level is equivalent to common knowledge in a higher–level state set.

My assumptions are consistent with this view. I assume that the information structure is common knowledge. However, I assume common knowledge at a higher level than is usually analyzed. In most bargaining models, it is assumed that a party is either informed or uninformed about the payoffs to various possible agreements and that which parties are informed is common knowledge.[3] By contrast, I assume in Section 3.1, for example, that the seller is informed, that there is a probability that the buyer is informed, and that these facts are common knowledge.

Related Literature. While there is a large literature on common knowledge (see, *e.g.*, Aumann [1976], Geanakoplos and Polemarchakis [1982], and Mertens and Zamir [1985]), only a few authors have explored the implications of a lack of common knowledge. Milgrom and Roberts [1982, Appendix B] discuss the possibility that lack of common knowledge could generate predation in Selten's [1978] Chain–Store game. Aumann [1988] gives an example in which the paradox of Rosenthal's [1981] centipede game is resolved by allowing for the possibility that rationality is not common knowledge. Rubinstein [1989] uses the coordinated attack problem to show that "almost common knowledge" is quite different from common knowledge. Neyman [1989] analyzes the effect of relaxing common knowledge assumptions in finitely repeated games.

3. EXOGENOUS INFORMATION

The basic game I consider has one seller, one buyer, and a single indivisible unit of a good. The seller proposes a price to the buyer. The buyer can then accept or reject. If he accepts, they trade at the proposed price. The payoff to the seller is the price minus his cost; the payoff to the buyer is the value of the good to him minus the price. If the buyer rejects the offer, the game ends and each player gets a payoff of zero. I assume throughout that the buyer's valuation is always larger than the seller's costs, so that trade is efficient. I also assume that the seller knows his costs and the buyer's valuation.

Formally, there is a finite set of possible states of the world, $S \subset [0,1]$, with $0, 1 \in S$. The buyer's valuation is given by a function $v : [0,1] \to \mathbf{R}_+$ and the seller's costs by $c : [0,1] \to \mathbf{R}_+$. Of course, these functions are only relevant at points in S, but, for reasons that will be clear shortly,

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[2] See Brandenburger and Dekel [1985], Tan and Werlang [1985], and Gilboa [1988] for various formalizations of this argument. See also Mertens and Zamir [1985].

[3] See, for example, the papers in the recent *Journal of Economic Theory* symposium [1989].

it is convenient to define them on a broader domain. Without loss of generality, I assume that $v(s)$ is weakly increasing in s with $v(1) > v(0)$. Trade is always efficient in the sense that there is a $k > 0$ such that $v(s) - c(s) \geq k$ for all $s \in [0,1]$. Throughout, I assume that the seller knows the true state of the world. The buyer has a prior probability distribution f over S such that $f(s) > 0$ for all $s \in S$.[4]

While assuming S is finite simplifies much of the mathematics, it is inconvenient in that some results depend on how far apart the values of v and c are across states. For this reason, I sometimes assume that the largest distance between consecutive values of $v(s)$ or $c(s)$ for $s \in S$ is $\psi(n)$, where n, the number of states, is "large" and $\psi(n) \to 0$ as $n \to \infty$. When this assumption is used, I state it more succinctly by simply saying that n is large. It is only when I assume that n is large that the properties of $v(\cdot)$ and $c(\cdot)$ over all of $[0,1]$ are relevant.

I analyze the pure strategy sequential equilibria of variations on this game. Intuitively, a sequential equilibrium specifies strategies and beliefs for each player. Each player's strategy is required to be a best reply to the other player's strategy given his beliefs. Each player's beliefs are required to be consistent with the other player's strategy and Bayes' rule where applicable. A well known problem is the fact that the sequential equilibrium concept does not provide useful restrictions on beliefs off the equilibrium path. This allows players to "threaten with beliefs"—that is, they can adopt intuitively implausible beliefs off the equilibrium path which justify punitive actions in response to the deviation. The ability to punish deviations arbitrarily greatly enlarges the set of equilibria. In this context, this problem is especially troubling since my intent is to compare equilibria under various information structures. Naturally, I do not want the comparisons to hinge on what kinds of uncertainty allow the buyer more flexibility to threaten with his beliefs.

Consequently, I use refinements of sequential equilibria which eliminate some forms of threatening with beliefs. In Section 4, I focus on sequential equilibria in which no player ever uses a dominated strategy. This eliminates the buyer's ability to threaten to use a dominated strategy which punishes the seller. In Section 3.2, I use much stronger notion, namely perfect sequential equilibrium (see Grossman and Perry [1986]). Intuitively, this equilibrium concept eliminates sequential equilibria in which off the equilibrium path beliefs do not always use "sensible explanations" for the deviation when such explanations exist.

3.1 Common Knowledge

The simplest case is where whether or not the buyer is informed is common knowledge. When

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[4] My assumptions on V and C are consistent with any degree of correlation between V and C as well as independence. Vincent [1989] shows that correlation between V and C can have strong consequences in bargaining models.

the buyer is informed, the game is as follows. First, Nature chooses s according to the probability distribution f. The seller observes Nature's choice and chooses an offer of a price $p(s)$. The buyer observes s and $p(s)$ and then can either accept or reject. The payoffs to the buyer and seller are $(p(s) - c(s), v(s) - p(s))$ if the buyer accepts; the payoffs if he rejects are $(0,0)$. When the buyer is uninformed, the game is the same except the buyer only observes $p(s)$. I refer to these games as G_i and G_u respectively.

If the buyer is informed, the fact that the seller has all the bargaining power implies that he gets all the gains from trade. More specifically, it is well-known that the unique sequential equilibrium of G_i has the seller choosing $p(s) = v(s)$ and the buyer accepting any $p \leq v(s)$. Not surprisingly, if the buyer does not know the value of the good and this fact is common knowledge, this informational disadvantage does not help him. It is true that there is no longer an equilibrium in which the seller offers $p(s) = v(s)$ and the buyer accepts. If the buyer would accept any such offer, the seller only offers the largest value of $v(s)$. However, if we rule out sequential equilibria which rely on "threatening with beliefs," the buyer's expected payoff is necessarily zero. More specifically, we have the following result.[5]

Theorem 1. *The buyer's expected payoff is zero in every perfect sequential equilibrium of G_u.*

In a perfect sequential equilibrium of G_u, the seller offers a price satisfying

$$p = \mathrm{E}[v(s) \mid p \geq c(s)]$$

if this at least covers his costs. Clearly, this makes the buyer's expected payoff zero. Intuitively, the seller wishes to set the price as high as possible and so sets it at the point where the buyer is just indifferent between accepting and rejecting the price, taking account of the information revealed by the fact that the seller is willing to trade at that price. Since the buyer is indifferent, it must be true that his expected payoff is zero—that is, he gets none of the gains from trade.

In short, if what the buyer knows is common knowledge, the seller's bargaining power makes the buyer's expected payoff is zero whether he knows his valuation or not. However, if the seller does not know whether the buyer knows his valuation, this reduces his bargaining power, potentially enabling the buyer to get a strictly positive payoff.

3.2 Equilibrium without Common Knowledge

Suppose the seller does not know the buyer's prior beliefs regarding the state. More specifically, Nature chooses s according to a nondegenerate probability distribution f. The seller observes

[5] The proof, messy but not difficult, is omitted. It is available from the author by request.

Nature's choice and offers a price $p(s)$. With probability $q \in (0,1)$, the buyer observes s and $p(s)$. Otherwise, the buyer observes only $p(s)$. Finally, the buyer can accept or reject with payoffs as above. Let G_q denote this game. The structure of G_q—including the value of q—is assumed to be common knowlege.

Clearly, if the buyer is informed, he accepts iff $p \leq v(s)$. Because of this, if q is close to 1, the strategies from the complete information case (appropriately modified) are still an equilibrium. To see this, suppose the seller offers $p(s) = v(s)$. The best reply for the buyer, if not informed, is to always accept. Is the seller's strategy a best reply? It is easy to see that his strategy is optimal iff for all s

$$v(s) - c(s) \geq (1-q)[v(1) - c(s)].$$

(Recall that $v(s)$ is increasing and that 1 is the largest value of s.) Rearranging:

$$q \geq q^*(s) = \frac{v(1) - v(s)}{v(1) - c(s)}.$$

Let q^* denote the largest value of $q^*(s)$ for $s \in S$. It is easy to use the assumptions on v and c to show that $q^* \in (0,1)$. Hence if q is close enough to 1, there is a sequential equilibrium in which the buyer receives no surplus. In fact, every perfect sequential equilibrium for $q \geq q^*$ has this property.

If $q \in (0, q^*)$, though, it simply is not credible for the seller to price at the buyer's valuation. If the buyer expects the seller to do so, the seller deviates to charging $v(1)$. Because of this, the informed buyer necessarily receives a strictly positive payoff. This also may enable the uninformed buyer to gain part of the surplus. However, consider what happens as q becomes small. Intuitively, we would expect the analysis to parallel the case where q is large—that is, for small enough q, the seller is essentially unconcerned with the behavior of the informed buyer. If q is sufficiently small, then, the seller always offers the highest price the uninformed buyer will accept. As in Theorem 1, this leaves the uninformed buyer indifferent between accepting and rejecting, giving him an expected payoff of zero. Unfortunately, the analysis is not quite so simple, since some types of sellers would earn negative profits at this price. In G_u, these types make no offer. In this game, though, these types offer higher prices in the hope that the buyer is informed. In part because of this, the analysis of the case where q is close to zero is more complex than the case where q is close to one. However, the result is the same: if the seller is almost sure of what the buyer knows, the buyer's payoff is close to zero. These observations are summarized in the following theorem, proved in the Appendix.[6]

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[6] Perfect sequential equilibria often do not exist. While general existence results for this game are difficult, it is easy to show existence for a variety of special cases.

Theorem 2. *The buyer's expected payoff is zero for all $q \geq q^*$ in every perfect sequential equilibrium of G_q. For $q \in (0, q^*)$, the informed buyer's expected payoff is strictly positive if n is "large," though the uninformed buyer's payoff can be zero or strictly positive. However, as $q \downarrow 0$, the buyer's expected payoff in any sequence of perfect sequential equilibria of G_q goes to zero.*

In short, if q is far from zero and one, the buyer, at least if he is informed, earns a positive payoff. However, for q close to zero or one, the buyer's expected payoff is close to zero. In this sense, the outcome is continuous as a function of the seller's knowledge about the buyer's knowledge.

4. ENDOGENOUS INFORMATION

When the buyer can choose whether or not he knows his valuation, this continuity disappears. To be specific, suppose that after the seller makes his offer, the buyer can learn his valuation at a cost of $\delta \geq 0$.[7] If $\delta = 0$, the outcome is the same as the complete information world: the seller offers a price equal to the buyer's valuation, the buyer costlessly learns his valuation, and accepts. However, the outcome changes radically for $\delta > 0$ but small. We will see that the buyer's payoff is bounded away from 0 for all sufficiently small $\delta > 0$. In particular, the limit as $\delta \downarrow 0$ of the infimum of the buyer's payoff is strictly positive and can be quite large.

Formally, the game considered in this section is as follows. First, Nature chooses s according to the probability distribution $f(s)$. The seller observes s and offers a price $p(s)$. The buyer observes only $p(s)$ and then can choose to observe s. Then the buyer decides whether or not to accept. If the buyer does not observe s, the payoffs are exactly as in the previous section. If the buyer does observe s, the seller's payoff is as above, while the buyer's payoff is reduced by δ. I refer to this game as G_δ.

The case where $\delta = 0$ is quite simple. It is easy to see that not learning s given any $p \in (v(0), v(1))$ is a dominated strategy when $\delta = 0$. Hence in every sequential equilibrium where neither player uses a dominated strategy, the buyer learns s. Knowing this, the seller always sets $p(s) = v(s)$ and the buyer accepts. Hence we have the following theorem.

Theorem 3. *If $\delta = 0$, G_δ has a unique sequential equilibrium without use of dominated strategies in which the seller offers $p(s) = v(s)$, the buyer learns s whenever $p \in (v(0), v(1))$, and the buyer accepts any $p \leq v(s)$. Hence the buyer's expected payoff is zero.*

Suppose instead that δ is strictly positive but small. In this case, the strategies of Theorem 3

[7] The results are entirely unaffected by allowing the buyer the option of learning his valuation either before or after the seller's offer.

cannot possibly be an equilibrium. Since the seller's offer reveals $v(s)$ perfectly, the buyer will not pay any strictly positive amount to learn s. More generally, we see that it is no longer a dominated strategy to not learn s. Whenever the seller's strategy is such that there is a zero probability that the price is above $v(s)$, then the buyer's strict best response is to accept the offer without learning s. Hence it becomes credible for the buyer to not learn s, preventing the seller from extracting all the gains from trade.

To state the implications of this, for any $\delta > 0$, let $p(\delta)$ be the p which solves

$$\delta = \sum_{s \in \{s' | p \geq v(s')\}} f(s)[p - v(s)].$$

For any p, let $s^*(p)$ be the largest s such that $p > c(s')$ for all $s' \leq s$.

Theorem 4. *For any $\delta > 0$, every sequential equilibrium of G_δ gives the buyer an expected payoff of at least*

$$\inf_{p \in [v(0), p(\delta)]} \sum_{s \leq s^*(p)} f(s)[v(s) - p].$$

The proof of this result, contained in the Appendix, is not difficult. I will say that a price is automatically rejected (accepted) in an equilibrium if the buyer responds to that offer by rejecting (accepting) it without learning s. Let p be the lowest price the seller offers in some equilibrium which is not automatically rejected. Clearly, no offer above p is automatically accepted—if it were, the seller would never offer p. Also, p must be automatically accepted. If it is not, then it must be true that the buyer responds by learning the state. Since this is costly, the buyer would only do this if there exists an s such that $p(s) = p$ and $v(s) < p$. Hence it must be true that $v(0) < p$. But then, since every offer is either automatically rejected or leads the buyer to learn s, we see that when $s = 0$, the seller does not trade. But this cannot be an equilibrium since the seller could do better in this case by offering $v(0) - \epsilon$ when $s = 0$. (Recall that $v(0) > c(0)$.) It is not hard to show that if p is automatically accepted, it must be smaller than $p(\delta)$. If it were larger, the buyer would be better off learning s and rejecting the offer when $p > v(s)$. Finally, as shown in the Appendix, the seller offers p at least when $s \leq s^*(p)$. Since the buyer's expected payoff conditional on any price must be nonnegative, the lower bound given in Theorem 4 follows.[8]

The contrast between Theorems 2 and 4 is striking. When information is exogenous, the seller's uncertainty about the buyer's knowledge helps the informed buyer if q is small enough, but

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[8] It is not difficult to construct examples of pure strategy sequential equilibria with undominated strategies for this game, so a comparison of Theorems 3 and 4 is not vacuous. One can often achieve the lower bound on the buyer's expected payoff.

may never help the uninformed buyer. When information is endogenous, the situation is reversed: the lower bound on the buyer's expected payoff is based on his payoff when he chooses to remain ignorant. It is easy to construct examples in which the buyer's expected payoff is strictly positive if and only if the seller's offer leads him to remain uninformed.

The contrast between Theorems 3 and 4 is also surprising. First, notice that the lower bound on the buyer's expected payoff immediately implies the discontinuity referred to above. As $\delta \downarrow 0$, $p(\delta) \downarrow v(0)$, so the lower bound converges to

$$\sum_{s \leq s^*(v(0))} f(s)[v(s) - v(0)].$$

If there is at least one $s' > 0$ such that $c(s') < v(0)$, then $s^*(v(0)) \geq v'$. (Note that this must be true if n is sufficiently large.) Hence the limit as $\delta \downarrow 0$ of the buyer's payoff is at least $f(s')[v(s') - v(0)] > 0$. Thus the buyer's payoff as a function of δ is discontinous at $\delta = 0$.

Notice also that this lower bound can be quite strong. For example, suppose that $c(s) < v(0)$ for all s. (This must hold if the buyer's valuation is independent of the seller's costs by the assumption that trade is always efficient.) In this case, $s^*(p) = 1$ for any $p \geq v(0)$. Thus the lower bound is just $\mathrm{E}(v) - p(\delta)$. Hence as $\delta \downarrow 0$, the buyer's payoff converges to at least $\mathrm{E}(v) - v(0)$. Of course, the buyer can never get a larger payoff than this since the seller never offers a lower price than $v(0)$. Hence the limit of the buyer's payoff is exactly $\mathrm{E}(v) - v(0)$. This holds even if $c(s) = v(0) - \epsilon$ for all s. In this case, the seller's payoff is ϵ, so that the buyer receives almost all the surplus!

Why is the equilibrium outcome discontinuous in δ at $\delta = 0$?[9] Intuitively, when the buyer has the ability to learn his valuation, this prevents the seller from setting the price much above the buyer's valuation. As $\delta \downarrow 0$, the probability that the price strictly exceeds the buyer's valuation must go to zero. At the same time, as long as $\delta > 0$, it is not an equilibrium for the seller to set the price equal to the buyer's valuation. If he does so, the buyer will never pay any strictly positive amount to learn his valuation since the seller's price reveals it to him. But then the seller has an incentive to deviate to the highest possible price. Even as $\delta \downarrow 0$, this effect prevents the seller from increasing the price up to the buyer's valuation.

[9] This discontinuity does not occur when the equilibrium notion is sequential equilibria since the sequential equilibrium correspondence is upper semicontinuous in payoffs (Kreps and Wilson [1982b]). However, the sequential equilibrium which is the limit as $\delta \downarrow 0$ requires the buyer to not learn s in response to some price offers above $v(0)$. This strategy is dominated when $\delta = 0$, even though it is not when $\delta > 0$. As this illustrates, the correspondence giving sequential equilibria in which no player uses a dominated strategy is not upper semicontinuous.

In short, when the buyer's valuation is common knowledge or if the buyer's ignorance about his valuation is common knowledge, the buyer gets none of the gains from trade. If, however, the buyer's information or ignorance is not common knowledge, the buyer does better. Even when the cost of learning his valuation is infinitesimal, the fact that the seller cannot be certain of what the buyer knows prevents the seller from getting all the surplus. Intuitively, the lack of common knowledge gives the buyer enough of an informational advantage to guarantee himself some of the surplus. This is analogous to the way the buyer gets some of the gains from trade in bargaining games where he knows his valuation and the seller does not.

It is worth noting the importance of the timing assumed in these results. The key to the result is the way the information conveyed by the seller's offer interacts with the buyer's decision to learn the state. If, for example, the buyer could learn the state before the seller's offer but not after the offer, the discontinuity would not occur. (If he has the option of learning the state either before or after, my results are unaffected.) Similarly, this discontinuity would not be present if we replace the bargaining model used here with a k–double auction (see, for example, Satterthwaite and Williams [1989]). In a k–double auction, the buyer and seller *simultaneously* make price offers, say p_b and p_s. If $p_s > p_b$, no trade occurs. If $p_s \leq p_b$, trade takes place at a price of $kp_b + (1-k)p_s$ where $k \in [0,1]$. The simultaneity of the procedure means that the buyer would have to make his decision about learning s before getting to see the seller's offer. Hence the discontinuity in δ at $\delta = 0$ would not arise.

5. APPENDIX

5.1 Proof of Theorem 2

Lemma 1. *Let $\bar{p}$ be the largest price the uninformed buyer accepts in some perfect sequential equilibrium. Then the seller always offers either $v(s)$ or $\bar{p}$. If $v(s) < \bar{p}$, the seller offers $v(s)$ if*

$$v(s) - c(s) > (1-q)[\bar{p} - c(s)]$$

and offers $\bar{p}$ if the reverse strict inequality holds. If $v(s) > \bar{p}$, he offers $v(s)$ if

$$q[v(s) - c(s)] > \bar{p} - c(s)$$

and $\bar{p}$ if the reverse strict inequality holds. If the buyer is informed, he accepts any $p \leq v(s)$. If he is uninformed, he accepts any value of $v(s) \leq \bar{p}$ and accepts $\bar{p}$.

Proof: Consider any s with $v(s) < \bar{p}$. Suppose that in equilibrium this type of seller only sells to the informed buyer. Then it must be true that $p(s) \leq v(s)$. Clearly, though, we cannot have $p(s) < v(s)$ since the seller could strictly increase the price and be better off as the informed buyer would continue to purchase. Hence if he sells only to the informed buyer, $p(s) = v(s)$. If he sells only to the uninformed buyer, we must have $p(s) = \bar{p}$, since this is the largest price the uninformed buyer accepts. Finally, if he sells to both types of buyer, we must have $p(s) \leq v(s)$. Suppose $p(s) < v(s)$. Consider the deviation to $v(s) - \epsilon$ where $\epsilon > 0$ but is small enough that $v(s) - \epsilon > p(s)$ and $v(s) - \epsilon > v(s')$ for all s' such that $v(s') < v(s)$. Clearly, the informed buyer accepts this offer. Suppose the uninformed buyer accepts. Then the seller certainly deviates in state s. Notice, however, that there is no state s' with $v(s') < v(s)$ in which the seller would deviate. Since $v(s) - \epsilon > v(s')$, if the seller in state s' deviated, he would only sell to the uninformed buyer. By assumption, though, $v(s) < \bar{p}$, so the seller in state s' could do better by selling to the uninformed buyer at $\bar{p}$. Hence for any s' such that the seller deviates, we must have $v(s') \geq v(s)$. Hence it is optimal for the uninformed buyer to accept the offer, breaking the proposed equilibrium. Therefore, if the seller does sell to both types in a state with $v(s) < \bar{p}$, we must have $p(s) = v(s)$. The same argument can be used to show that if $v(s) < \bar{p}$, then the seller in state s will not sell only to the informed buyer at $p(s) = v(s)$. If he were only able to sell to the informed buyer at $p = v(s)$, he could deviate down ϵ and attract the uninformed buyer, leading to a higher payoff.

Hence for states such that $v(s) < \bar{p}$, the seller either sells to both types of buyers at $v(s)$ or only to the uninformed buyer at $\bar{p}$. Essentially the same argument establishes that if the payoff to the seller from selling to both types at $v(s)$ is larger than the payoff to selling only to the uninformed at $\bar{p}$, then in equilibrium, the seller offers $v(s)$ and both types do accept. In other words, the seller has both options in equilibrium and chooses the one which yields the higher profits. Thus for states such that $v(s) < \bar{p}$, $p(s) = v(s)$ if

$$v(s) - c(s) > (1 - q)[\bar{p} - c(s)]$$

and $p(s) = \bar{p}$ if the opposite strict inequality holds, as stated in the lemma.

Suppose, then, that $v(s) > \bar{p}$. Clearly, then, the seller either sells to both types of buyer at $\bar{p}$ or sells only to the informed buyer. If he does the latter, we must have $p(s) = v(s)$. Hence, again, the statement of the lemma is verified. Finally, suppose that $v(s) = \bar{p}$. The seller can sell to both types at $\bar{p}$ or at a lower price. At any higher price, he cannot sell to either type. Clearly, $p(s) = \bar{p}$ is optimal. ▌

Lemma 2. *If $q \geq q^*$, then the buyer's expected payoff is zero in every perfect sequential equilibrium.*

Proof: Suppose, contrary to the lemma, that there is a perfect sequential equilibrium in which the buyer's payoff is strictly positive. Let $\bar{p}$ be the highest price the uninformed buyer accepts. By Lemma 1, the seller always offers either $v(s)$ or $\bar{p}$. Hence if the buyer has a strictly positive expected payoff, it must be true that there is some state s such that $p(s) = \bar{p} < v(s)$. Clearly, this means that we must have $\bar{p} < v(1)$. But, by the definition of q^*, the fact that $q \geq q^*$ implies that every seller prefers to offer $v(s)$, a contradiction. ▌

Lemma 3. *If $q \in (0, q^*)$, then the buyer's expected payoff if he is informed is strictly positive if n is sufficiently large.*

Proof: Suppose not. Since the informed buyer's expected payoff cannot be strictly negative, it must be zero. Again, let $\bar{p}$ be the largest price the uninformed buyer accepts. By Lemma 1, either $p(s) = v(s)$ or $p(s) = \bar{p}$. If the informed buyer's payoff is zero, either $p(s) = v(s)$ for all s or $\bar{p} \geq v(s)$ for all s such that $p(s) = \bar{p}$. Clearly, if $\bar{p} > v(s)$ for some s such that $p(s) = \bar{p}$, the uninformed buyer rejects $\bar{p}$, a contradiction. Hence for every s such that $p(s) = \bar{p}$, we have $v(s) = \bar{p}$. But then $p(s) = v(s)$ for all s. Since $q < q^*$, this implies $\bar{p} < v(1)$.

It is easy to see that $\bar{p} \geq v(0)$. Otherwise, the seller in state 0 could deviate to $v(0) - \epsilon$ and be strictly better off (as this must be accepted by both the informed and uninformed buyer). Let $\bar{s}$ denote the largest s such that $v(s) \leq \bar{p}$ and let $\bar{s}_+$ denote the smallest state with $v(s) > v(\bar{s})$. Since, by definition, $v(\bar{s}_+) > \bar{p}$, the uninformed buyer does not accept a price of $v(\bar{s}_+)$. Hence the seller in state $\bar{s}_+$ sells only to the informed buyer. He prefers this to deviating down to a price of $\bar{p}$ iff

$$(1) \qquad q[v(\bar{s}_+) - c(\bar{s}_+)] \geq \bar{p} - c(\bar{s}_+) \geq v(\bar{s}) - c(\bar{s}_+).$$

Recall that the maximum distance between consecutive values of $v(s)$ or of $c(s)$ is $\psi(n)$. Together with (1), this implies

$$\psi(n) \geq (1 - q)[v(\bar{s}) - c(\bar{s})] \geq (1 - q^*)k$$

for all $q \leq q^*$. (Recall that $v(s) - c(s) \geq k$ for all $s \in [0, 1]$, where $k > 0$.) Hence, since $\psi(n) \to 0$ as $n \to \infty$, we can choose n large enough that (1) cannot hold for any $q \in (0, q^*)$. ▌

Lemma 3 shows that when $q \in (0, q^*)$ and n is large, we cannot have an equilibrium in which $p(s) = v(s)$ for all s. Since we cannot have $p(s) \geq v(s)$ for all s with a strict inequality for some s, this implies that the informed buyer earns a strictly positive expected payoff. This does not, however, guarantee that the uninformed buyer earns a strictly positive payoff. I now show that the uninformed buyer's payoff converges to 0 as $q \downarrow 0$ for every sequence of perfect sequential equilibria. Suppose not. Let $p_q(s)$ denote the seller's strategy as a function of q along this sequence of equilibria. Let $\bar{p}_q$ denote the largest price the uninformed buyer will accept as a function of q. Then

$$(2) \qquad \lim_{q \downarrow 0} \sum_{s \in \hat{S}_q} [v(s) - \bar{p}_q] f(s) > 0$$

where $\hat{S}_q = \{s \mid p_q(s) = \bar{p}_q\}$. A necessary condition for (2) is

$$\bar{p}_q < \mathrm{E}[v(s) \mid s \in \hat{S}_q]$$

for all q sufficiently small. Suppose there is no $s' \notin \hat{S}_q$ with $v(s') \leq \bar{p}_q$ such that the seller is indifferent between charging $v(s')$ and $\bar{p}_q$. Then there is an $\epsilon > 0$ such that no seller with $s' \notin \hat{S}_q$ and $v(s') \leq \bar{p}_q$ would deviate to $\bar{p}_q + \epsilon$ if the uninformed buyer would accept this price, but for every $s \in \hat{S}_q$, the seller would deviate. Let $\hat{S}_q^+$ denote the set of s such that the seller deviates. Clearly, $\hat{S}_q \subseteq \hat{S}_q^+$ and $v(s) > \bar{p}_q + \epsilon$ for all $s \in \hat{S}_q^+ \setminus \hat{S}_q$. Hence we can then make ϵ small enough that

$$\mathrm{E}[v(s) \mid s \in \hat{S}_q^+] > \bar{p}_q + \epsilon,$$

so the uninformed buyer's best response is to accept, breaking the proposed equilibrium.

Hence there must be some $s' \notin \hat{S}_q$ with $v(s') \leq \bar{p}_q$ such that the seller is indifferent between $v(s')$ and $\bar{p}_q$. Let S_q^I denote the set of such s. The argument above can easily be extended to show that we must have

(3) $$\mathrm{E}[v(s) \mid s \in \hat{S}_q \cup S_q^I] \leq \bar{p}_q.$$

By definition, $s \in S_q^I$ iff

$$v(s) - c(s) = (1-q)[\bar{p}_q - c(s)].$$

Clearly, then, for any $\epsilon > 0$, we can choose q small enough that we will have $\bar{p}_q - v(s) < \epsilon$ for all $s \in S_q^I$. This implies that there is a $\bar{q} > 0$ and a unique value of $v(s)$, say v^*, such that all $s \in S_q^I$ for $q \leq \bar{q}$ have $v(s) = v^*$.

For $q < \bar{q}$, then, (3) implies

$$\bar{p}_q \geq \frac{\Pr[s \in \hat{S}_q]}{\Pr[s \in \hat{S}_q \cup S_q^I]} \mathrm{E}[v(s) \mid s \in \hat{S}_q] + \frac{\Pr[s \in S_q^I \setminus \hat{S}_q]}{\Pr[s \in \hat{S}_q \cup S_q^I]} v^*$$

which can be rewritten as

$$\Pr[s \in \hat{S}_q \cup S_q^I](\bar{p}_q - v^*) > \Pr[s \in \hat{S}_q](\mathrm{E}[v(s) \mid s \in \hat{S}_q] - v^*) \geq 0.$$

But since $\lim_{q \downarrow 0} \bar{p}_q = v^*$, the left–hand side goes to zero, so the middle term must as well. By (2),

$$v^* = \lim_{q \downarrow 0} \bar{p}_q < \lim_{q \downarrow 0} \mathrm{E}[v(s) \mid s \in \hat{S}_q],$$

so we must have

$$\lim_{q \downarrow 0} \Pr[s \in \hat{S}_q] = 0.$$

However, this contradicts (2). ▌

5.2 Proof of Theorem 4.

Fix a sequential equilibrium and let $p_1 < \ldots < p_k$ be the set of prices offered by the seller in equilibrium which are not automatically rejected. As shown in the text, p_1 must be automatically accepted and for every $i > 1$, the buyer must respond to p_i by learning s.

Let $s^*(p_1)$ be the largest s such that $p_1 > c(s')$ for all $s' \leq s$. (Let $s^*(p_1) = 1$ if $p_1 > c(s)$ for all s.) Suppose that $p(s) \neq p_1$ for some $s \leq s^*(p_1)$. Let $\tilde{s}$ denote the smallest such s. Since the seller would receive a strictly positive payoff in state $\tilde{s}$ if he offered p_1, it must be true that he receives a larger payoff by offering p_i for some $i > 1$. Let $p(\tilde{s}) = \tilde{p}$. Since the buyer learns s in response to p_i for all $i > 1$, it must be true that $\tilde{p} \leq v(\tilde{s})$ as the seller would receive zero in state $\tilde{s}$ otherwise. By assumption, the seller offers p_1 for every $s < \tilde{s}$, so that $\tilde{s}$ must be the smallest s such that $p(s) = \tilde{p}$. But recall that $v(s)$ is weakly increasing in s. Hence the fact that $v(\tilde{s}) \geq \tilde{p}$ implies that $v(s) \geq \tilde{p}$ for every s such that $p(s) = \tilde{p}$. But then the fact that $\delta > 0$ implies that the buyer will not learn s in response to $\tilde{p}$, a contradiction. Hence the seller offers p_1 for every $s \leq s^*(p_1)$.

Let $S(p_1)$ denote the set of s such that $p(s) = p_1$. Since the buyer does not learn s in response to p_1, it must be true that

$$\sum_{s \in S(p_1)} [v(s) - p_1] f(s) \geq \sum_{s \in S(p_1)} \max[v(s) - p_1, 0] f(s) - \delta$$

or, letting $S^-(p_1)$ denote the set of $s \in S(p_1)$ with $v(s) \leq p_1$,

$$\delta \geq \sum_{s \in S^-(p_1)} [p_1 - v(s)] f(s). \tag{4}$$

By definition, $c(s^*(p_1)) > p_1$. Since $v(s) > c(s)$ for all s, $v(s^*(p_1)) > p_1$. Since $v(s)$ is weakly increasing, then, every s with $v(s) \leq p_1$ is smaller than $s^*(p_1)$. Since every such s is in $S(p_1)$, $S^-(p_1)$ is just the set of s such that $v(s) \leq p_1$. So we can rewrite (4) as

$$\delta \geq \sum_{s \in \{s' | p_1 \geq v(s')\}} f(s)[p_1 - v(s)]. \tag{5}$$

Let $p(\delta)$ be the value of p_1 such that this holds with equality. It is easy to show that a unique $p(\delta)$ exists for every $\delta > 0$. Since the right–hand side of (5) is increasing in p_1, (5) implies $p_1 \leq p(\delta)$.

Summarizing, there is a $p_1 \in [v(0), p(\delta)]$ such that p_1 is offered at least when $s \leq s^*(p_1)$. If the seller offers p_1 in some state $s > s^*(p_1)$, it will necessarily be true that $v(s) > p_1$ since $v(s)$ is increasing and $v(s^*(p_1)) > p_1$. Hence a lower bound on the buyer's expected payoff can be constructed by supposing that the buyer receives a payoff of zero in every state $s \in S(p_1)$ such that $s > s^*(p_1)$. Furthermore, conditional on any price, the buyer's expected payoff is nonnegative. Thus we can construct a lower bound by assuming that the buyer's payoff is zero when any price other than p_1 is offered. Hence a lower bound on the buyer's expected payoff is

$$\inf_{p_1 \in [v(0), p(\delta)]} \sum_{s \leq s^*(p_1)} [v(s) - p_1] f(s). \quad \blacksquare$$

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