Topological Reasoning and The Logic of Knowledge

(Preliminary Report)

Lawrence S. Moss
Department of Mathematics
Indiana University
Bloomington, IN 47405

Rohit Parikh
Departments of Computer Science,
Mathematics and Philosophy
CUNY Graduate Center
New York, NY 10036
Department of Computer Science
Brooklyn College of CUNY

1 Introduction

What are fields of mathematics, such as probability theory, point-set topology, and combinatorics, *about*? When asked this, a mathematician is likely to answer that the field is about various mathematical concepts, or about the consequences of some axioms or other. Although this answer would be adequate for many purposes, it misses a deeper answer that areas of mathematics can be seen as repositories for our intuitions about several aspects of ordinary life. For example, combinatorics can be seen as just the mathematical home for intuitions about activities like *counting* and *arranging*. General topology can be seen as the home for intuitions about *closeness*.

The point of this paper is to suggest that simple aspects of topological reasoning are also connected with special-purpose logics of *knowledge* . Traditionally ([HM], [PR], [CM]), knowledge is defined in terms of the notion of *view*. An individual has a view of the world or state and what that individual knows is whatever is true in all states which are compatible with its view. Thus, for instance, in distributed computing, the role of view is played by what a processor sees, i.e. its local history. Other notions that arise, like common knowledge, can then be defined in a natural way. Similarly, in Mathematical Economics, the notion of view is formalised through some *partition*. If the actual state is s, then the individual knows only that it is in the equivalence class of s, and this class can then be identified with its view.

However, in topology as well as in the theory of computation, a second notion, that of *effort* enters. Thus if I am at some point p and make a measurement, I will then discover that I am in some neighbourhood u of p, but not know exactly where. If I make my measurement finer, then u will shrink, say, to a smaller neighbourhood v. A similar consideration arises also in computation. If I am willing to compute for 10 steps, then I may discover that

$f(0) = 1$ and that $f(1) = 2$. However, the computation of $f(2)$ takes more than 10 steps and so I may not know whether I am computing the successor function or not. If, however, I am willing to invest 20 steps, then perhaps I will find that $f(2) = 7$ and so f is not the successor function after all.

Such a situation may also arise in ordinary life. Thus if a policeman is measuring the speed of passing cars, his knowledge is confined to the cars that are in his view, i.e. in this case, those cars that he can see. However, his knowledge of the speeds of the cars that he *can* see will depend on the accuracy of his measuring instrument. He can increase his knowledge *without* changing his view, by just using a more accurate measuring instrument. Nonetheless, his knowledge will generally be such that he can always improve it. This fact forces us to represent the situation using *two* modalities, one for knowledge, which is the usual K, and the other which depends on the effort, and this second modality will denoted by our familiar symbol $\Box$.

Our goal is to exhibit a formal system which can express simple topological reasoning. We would prefer the formal system not to be *too* expressive, for several reasons. First, we know that mathematical reasoning *may* be represented in a relatively strong system like first-order logic, but that this representation is not always suitable because first-order logic gives one the ability to make complicated assertions that were not seen to be relevant at the outset; that is, *because* it is universally applicable, its particular application in any setting may be unreasonably expressive. (When we move to higher-order logic, the situation is even worse.) Second, a less expressive system is likely to have a lower complexity. This would correspond to our intuition that certain kinds of reasoning in point-set topology are in some sense *easy*.

Naturally, in a computational context, the role of effort is played not by the topology, but by the amount of resource (time or space) devoted to the computation. The greater the resource devoted to the computation, the more one knows about the function being computed. We suspect that such an analogy based on knowledge theoretic considerations lies at the root of the analogies between descriptive set theory and in recursive function theory, analogies that have been noticed before, by Addison and others. Moreover, topological reasoning may also be compared to reasoning about approximations and about uncertainty. This is an important source for our ideas, connecting them with recent work in computer science (Vickers [Vi] and others).

A second motivation for our efforts is the intuition that topological reasoning does not arise in a linguistic vacuum. Instead, ordinary life already contains examples of situations where knowledge and effort combine; thus, there must exist hitherto unsuspected analogies which may be fruitful or enlightening to explore. A recent example is the theory of NP-completeness. The sense in which an NP set resembles an r.e. set or an open set is that it has similar knowledge properties. If our data is *in* the set, then it is *possible* for us to know this (even though it may not be *true* that we know it). With a co-NP-complete set the situation will be different assuming that our computational powers are limited.

Some readers may also recall the 'being drafted' analogy for r.e. sets. Being drafted is like being in an r.e. set in that, *if* you are drafted, you will know it because you will receive a draft notice. If you are *not* drafted, then you do not receive a non-draft notice, and therefore you do not know that you are not being drafted. In real life, of course we realise when we are 35 that we can no longer be drafted. This fact also has an analogy with recursive function theory since an r.e. set, for which we know a recursive bound on the amount of computation, will be recursive. Thus a knowledge theoretic account of elementary topology is likely also to make such analogies transparent.

Without wishing to sound immodest, we would like to point to a similar situation with non-standard analysis. While it is known from results of Kreisel and Parikh ([Kr], [Pa1]) that non-standard analysis is a *conservative extension* of standard analysis, nonetheless, it allows us to think in terms of infinitesimals, something that standard analysis does not permit us. As experience with 19th century analysis shows us, thinking in terms of infinitesimals is natural for us and thus a logical tool that *better* represents our thought processes may lead to insights or original results (see e.g. [BR]) that are more difficult to arrive at with a clumsier tool.

A final motivation for our work is the development of logical tools for visual reasoning, and with educational software that goes with this. Recently, Shin [Sh] gave a formal account of the use of Venn Diagrams as a tool in elementary reasoning. In a sense, her work is part of an ongoing rehabilitation of diagrammatic reasoning vis a vis purely symbolic manipulation. We believe that elementary topology should be amenable to a similar treatment, since in fact diagrams are as essential in point-set topology as they are in set theory. It is likely that the kind of logical tools we are developing will be useful in an analysis of how diagrammatic reasoning is useful in learning elementary topology.

2 Preliminaries

2.1 A Language and Its Semantics

Although we are primarily interested in the special-purpose reasoning found in topology, recursive function theory and elsewhere, we formulate our logical questions about a much larger class than the topological spaces.

Definition A **subset space** is a pair $\mathcal{X} = \langle X, \mathcal{O} \rangle$ where X is a set and $\mathcal{O}$ is a set of subsets of X. We assume that $X \in \mathcal{O}$, though this is really not necessary. $\mathcal{X}$ is **closed under intersection** if whenever $S, T \in \mathcal{O}$, $S \cap T \in \mathcal{O}$. We can similarly define the notion of closure under *union*.

The idea is that p is a point which represents the way the world is, and a set u such that $p \in u$ is an observation that can be made about p. Since we come to know the world through observation, we consider a logic whose formulas are statements about the process of observation. Later we shall exapand this language.

Definitions Let $\mathcal{A}$ be an arbitrary countable set of *atomic formulas.* $\mathcal{L}$ is the smallest set containing each $A \in \mathcal{A}$, and closed under the following formation rules: if $\phi, \psi \in \mathcal{L}$, then so are $\phi \wedge \psi$ and $\neg\phi$; if $\phi \in \mathcal{L}$, then $K\phi \in \mathcal{L}$ and $\Box\phi \in \mathcal{L}$.

Let $\mathcal{X}$ be a subset space. Thinking of $\mathcal{X}$ as a Kripke *frame*, we give the semantics of $\mathcal{L}$ relative to an interpretation of the atomic formulas. Such an interpretation is a map $i : \mathcal{A} \to \mathcal{P}X$. The pair $(\mathcal{X}, i)$ will be a *model*, and we let $\mathcal{M}$ denote models.

For $p \in X$ and $p \in u \in \mathcal{O}$, we define the **satisfaction relation** $\models_{\mathcal{M}}$ on $(X \times \mathcal{O}) \times \mathcal{L}$ by recursion on ϕ.

$p, u \models_{\mathcal{M}} A$	iff $p \in i(A)$
$p, u \models_{\mathcal{M}} \phi \wedge \psi$	if $p, u \models \phi$ and $p, u \models \psi$
$p, u \models_{\mathcal{M}} \neg\phi$	if $p, u \not\models_{\mathcal{M}} \phi$
$p, u \models_{\mathcal{M}} K\phi$	if for all $q \in u$, $q, u \models_{\mathcal{M}} \phi$
$p, u \models_{\mathcal{M}} \Box\phi$	if for all $v \in \mathcal{O}$ such that $p \in v \subseteq u$, $p, v \models_{\mathcal{M}} \phi$

We adopt two abbreviations: $L\phi$ means $\neg K \neg\phi$, and $\diamond\phi$ means $\neg\Box\neg\phi$. So $p, u \models_{\mathcal{M}} L\phi$ if there exists some $q \in u$ such that $q, u \models_{\mathcal{M}} \phi$, and $p, u \models_{\mathcal{M}} \diamond\phi$ if there exists $v \in \mathcal{O}$ such that $v \subseteq u$ and $p, v \models_{\mathcal{M}} \phi$.

We write $p, u \models \phi$ if $\mathcal{M}$ is clear from context. Finally, if $T \subseteq \mathcal{L}$, we write $T \models \phi$ if for all models $\mathcal{M}$, all $p \in X$, and all $u \in \mathcal{O}$, if $p, u \models \psi$ for each $\psi \in T$, then also $p, u \models \phi$.

Certain kinds of formula will have special interest for us. Given a model $\mathcal{M}$, and a formula ϕ, ϕ is *local in* $\mathcal{M}$ if for all p, u, v we have $p, u \models_{\mathcal{M}} \phi$ iff $p, v \models_{\mathcal{M}} \phi$. It is local if it is local in all $\mathcal{M}$. A formula ϕ is *persistent* in a model $\mathcal{M}$ if $\phi \to \Box\phi$ is valid in $\mathcal{M}$ and *persistent* if it is valid in every $\mathcal{M}$. Every formula of the form $K\Box\phi$ is persistent. Persistent formulas represent *reliable* knowledge and have a rather intuitionistic flavour. However, our logic is classical, since we are trying to represent certain knowledge theoretic ideas in a classical setting, rather than use an intuitionistic setting where such ideas would be *presupposed.* If the topology is discrete, then the only persistent formulas will be local. By contrast, with the trivial topology, only *tautologies* will tend to be persistent. Thus, for example, assuming that all boolean combinations of $i(A)$ and $i(B)$ are non-empty, then the only formulae involving A and B which are persistent will be the valid formulas. Note that when v is a subset of u, then every persistent formula satisfied by p, u is also satisfied by p, v confirming our intuition that refining from u to v increases knowledge.

Here are some observations which motivate our definitions: If $\mathcal{X}$ is indeed a topology, then a set $i(A)$ will be open iff every point in $i(A)$ has an open neighbourhood contained entirely in $i(A)$. This holds iff for all $p \in i(A)$, $p, X \models \diamond KA$. Thus $i(A)$ is *open* iff the formula $A \to \diamond KA$ is valid in the model. Dually, $i(A)$ is *closed* iff the formula $\Box LA \to A$ is valid in the model. The set $i(A)$ is *dense* iff the formula $\Box LA$ is valid and it is *nowhere dense* if the formula $\Box LK\neg A$ is valid. Finally, p belongs to the *boundary* of $i(A)$ iff $p, X \models \Box(LA \wedge L\neg A)$. So what we have is a logic in which topological concepts may be expressed, and which is not otherwise expressive. Note also that with the obvious definitions, r.e. subsets of the natural numbers will satisfy the same knowledge theoretic formula that opens do in a topological setting, and this, we believe, is the source of the similarity.

3 Completeness for Subset Spaces

Our main goal here is to sketch the proof of a completeness theorem for the relation $\models$. We shall follow techniques similar to those of [Ma], [Pa2], but the situation is vastly more complicated by the presence of two modalities and the need to create actual sets whose inclusion relation will be the accessibility relation of the Kripke structures needed.

We take the following as axioms:

Propositional Tautologies

$(A \to \Box A) \wedge (\neg A \to \Box \neg A)$, for $A \in \mathcal{A}$

$K\Box\phi \to \Box K\phi$

$K\phi \to (\phi \wedge KK\phi)$

$\phi \to KL\phi$

$\Box\phi \to (\phi \wedge \Box\Box\phi)$

$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

$K(\phi \to \psi) \to (K\phi \to K\psi)$

and we use the following rules of inference

$$\frac{\phi \to \psi,\ \phi}{\psi}\ MP$$

$$\frac{\phi}{K\phi}\ K\text{-necessitation} \qquad \frac{\phi}{\Box\phi}\ \Box\text{-necessitation}$$

Several of our axioms deserve discussion. Our semantics for atomic formulas A ignores the observation; that is, whether $p, u \models A$ depends only on p. This is the reason that $A \to \Box A$ is an axiom. Our framework is based on the intuition that the interpretations of atomic formulas are "absolute" in this sense. It would be possible to be more relativistic, of course. The other interesting axiom is the "cross axiom" $K\Box\phi \to \Box K\phi$. To see that it is sound, fix a subset space $\mathcal{X}$ and assume that $p, u \models K\Box\phi$; we claim that $p, u \models \Box K\phi$. To see this, let $v \subseteq u$. We need to see that $p, v \models K\phi$, so let $q \in v$. Then $q \in u$ also, so $q, u \models \Box\phi$. Therefore $q, v \models \phi$. Since q was arbitrary in u, we have $p, v \models K\phi$.

Lemma 1 (Soundness) *If $T \subseteq \mathcal{L}$ and $T \vdash \phi$, then $T \models \phi$.*

Section 4 contains an example of a proof in (an extension of) this logical system.

3.1 Properties of Theories

The proof of the Completeness Theorem uses maximal consistent subsets of $\mathcal{L}$, which we call *m-theories.*

Fix a language $\mathcal{L}$, and let $\mathcal{TH}$ be the set of m-theories in $\mathcal{L}$. In the sequel, we use U, V, etc., to denote m-theories. In order to prove that we have given a complete proof system, we need only show that for every m-theory T has a model there is a subset space $\mathcal{X}$, a point $x \in X$ and a subset $u \in \mathcal{O}$ such that $p, u \models_{\mathcal{M}} T$.

Definitions We define relations $\xrightarrow{L}$ and $\xrightarrow{\diamond}$ on m-theories by:
$U \xrightarrow{L} V$ iff whenever $\phi \in V$, $L\phi \in U$
$U \xrightarrow{\diamond} V$ iff whenever $\phi \in V$, $\diamond\phi \in U$.
Of course, the maximal consistency of m-theories give other characterizations. For example, $U \xrightarrow{L} V$ if whenever $K\phi \in U$, $\phi \in V$.

Further, define $U \xrightarrow{L\diamond} V$ if for some W, $U \xrightarrow{L} W \xrightarrow{\diamond} V$. And define $U \xrightarrow{\diamond L} V$ similarly.

Proposition 2 *The relation $\xrightarrow{L}$ is an equivalence relation. The relation $\xrightarrow{\diamond}$ is reflexive and transitive.*

The key fact about m-theories is the following fact:

Proposition 3 *If $U \xrightarrow{\diamond L} V$, then $U \xrightarrow{L\diamond} V$.*

The proof makes use of the cross axiom $K\Box\phi \rightarrow \Box K\phi$.

We might also mention a certain "pathology" concerning m-theories.

Proposition 4 *There are $U \neq V$ such that $U \xrightarrow{\diamond} V \xrightarrow{\diamond} U$.*

We know of no simple construction. The point of this proposition is that m-theories can be thought as the records of observations, given a point in the space and a set. The relation $\xrightarrow{\diamond}$ is a natural condition of refinement on observation records. Now if a space is sufficiently "homogeneous," then there might be many point-set pairs with the same observation record.

Theorem 5 *(Completeness For Subset Spaces) For all T and ϕ, $T \models \phi$ iff $T \vdash \phi$.*

3.2 The Strategy of The Proof

Here is a brief discussion of the strategy; the details are much too long to present here. The reader not interested in the proof should skip the rest of this section. We would like to build

(1) A set X containing a designated element x_0.

(2) A poset $\langle \mathsf{P}, \bot \rangle$ with least element $\bot$.

(3) A function $i : \mathsf{P} \to \mathcal{P}(X)$ such that $p \leq q$ iff $i(p) \supseteq i(q)$, and $i(\bot) = X$.

(4) A partial function $t : X \times \mathsf{P} \to \mathcal{TH}$ with the property that $t(x,p)$ is defined iff $x \in i(p)$. Furthermore, we require the following properties for all $p \in \mathsf{P}$, $x \in i(p)$, and $U \in \mathcal{TH}$:

   (a) $t(x,p) \xrightarrow{L} U$ iff for some $y \in i(p)$, $t(y,p) = U$.
   (b) $t(x,p) \xrightarrow{\diamond} U$ iff for some $q \geq p$ in P, $t(x,q) = U$.
   (c) $t(x_0, \bot) = T$, where T is the m-theory from above which we aim to model.

Suppose we have X, P, i, and t with these properties. Then we consider the subset space

$$\mathcal{X} \quad = \quad \langle X, \{i(p) : p \in \mathsf{P}\} \rangle.$$

To interpret the language P on $\mathcal{X}$, we stipulate that for atomic ϕ, $x, i(p) \models \phi$ iff $\phi \in t(x,p)$.

The poset P is an abstract description of the subset part of $\mathcal{X}$. The reason we build P is that in our later completeness constructions, we enrich P to a semi-lattice or lattice, and in this way incorporate information into our construction that would not otherwise be available. So our sketch here indicates how all of the completeness proofs go.

Lemma 6 (The Truth Lemma) *Assume conditions (1) – (4) for X, P, i, and t. Then for all $x \in X$ and all $p \in \mathsf{P}$ such that $x \in i(p)$,*

$$theory_{\mathcal{X}}(x, i(p)) \quad = \quad \{\phi : x, i(p) \models \phi\} \quad = \quad t(x,p)\ .$$

By the Truth Lemma and property (4c) above, $theory_{\mathcal{X}}(x_0, \bot) = T$. So if we could get X, P, i, and t, then T would have a model.

One immediate problem in trying to build everything is that the set $\mathcal{TH}$ is uncountable, so there will be too many (4) requirements. To get around this, we replace $\mathcal{TH}$ by a countable set $\mathcal{T}$ with the following properties:

(A) $T \in \mathcal{T}$

(B) If $U_0, V_0, V_1 \in \mathcal{T}$ and $U_0 \xrightarrow{\diamond} V_0 \xrightarrow{L} V_1$, then there is some $U_1 \in \mathcal{T}$ with $U_0 \xrightarrow{L} U_1 \xrightarrow{\diamond} V_1$.

(C) If $U \in \mathcal{T}$ and $L\phi \in U$, then for some $V \in \mathcal{T}$, $\phi \in V$ and $U \xrightarrow{L} V$.

(D) If $U \in \mathcal{T}$ and $\diamond\phi \in U$, then for some $V \in \mathcal{T}$, $\phi \in V$ and $U \xrightarrow{\diamond} V$.

Proposition 3 insures that a countable $\mathcal{T}$ exists with these properties.

We build X, P, i, and t to satisfy (1) –(3), and also the version of (4) where we replace $\mathcal{TH}$ by $\mathcal{T}$. The Truth Lemma again holds, and therefore T has a model.

The construction of X, P, i, and t is by recursion. That is, we build a sequence of approximations X_n, P_n, i_n, and t_n, and then take limits.

Fix two objects x_0 and $\perp$. The local and global properties that the construction will satisfy are as follows, where our numbering schemes are intended to be parallel to the one above for conditions (1)–(4):

(L1) X_n is a *finite* set containing x_0.

(L2) P_n is a finite poset with $\perp$ as minimum, and with the property that for each $p \in \mathsf{P}_n$, the lower set of p, $\{q \in \mathsf{P}_n : q \leq p\}$, is linearly ordered.

(L3) $i_n : \mathsf{P}_n \rightarrow \mathcal{P}(X_n)$ has the properties that $p \leq q$ iff $i_n(q) \subseteq i_n(p)$; also $i_n(\perp) = X_n$.

(L4) $t_n : X_n \times \mathsf{P}_n \rightarrow \mathcal{T}$ is a partial function with the property that $t_n(x,p)$ is defined iff $x \in i_n(p)$. Furthermore, we require the following properties for all $x \in X_n$ and $p \in \mathsf{P}_n$:

(a) If $x, y \in i_n(p)$, then $t_n(x,p) \xrightarrow{L} t_n(y,p)$.

(b) If $x \in i_n(q)$ and $q \geq p$, then $t_n(x,p) \xrightarrow{\diamond} t_n(x,q)$.

(c) $t_0(x_0, \perp) = T$.

(G1) $X_n \subseteq X_{n+1}$.

(G2) P_{n+1} is an end extension of P_n.

(G3) For all $p \in \mathsf{P}$, $i_{n+1}(p) \cap X_n = i_n(p)$.

(G4) The restriction of t_{n+1} to $X_n \times \mathsf{P}_n$ is t_n.

Finally, our construction has some overall requirements:

(R4a) If $t_n(x,p) \xrightarrow{L} U$ and $U \in \mathcal{T}$, then for some $m > n$, there is some $y \in i_m(p)$ such that $t_m(y,p) = U$.

(R4b) If $t_n(x,p) \xrightarrow{\diamond} U$ and $U \in \mathcal{T}$, then for some $m > n$, there is some $q \geq p$ in P_m such that $t_m(x,q) = U$.

Suppose we build X_n, P_n, i_n, and t_n in accordance with the (L), (G), and (R) requirements. Let $X = \bigcup_n X_n$, and let P be the limit of the posets P_n. Let i be defined by $i(p) = \bigcup_{n>m} i_{n+1}(p)$, where m is the least number such that $p \in X_m$. Finally, let $t(x,p) = t_n(x,p)$, where n is any number such that $t_n(x,p)$ is defined.

Proposition 7 *If X_n, P_n, i_n, and t_n satisfy the (L), (G), and (R) requirements, then X, P, i, and t as defined above satisfy conditions (1) – (4).*

Having come this far, we omit the actual details of the construction. Most of the complication comes from the bookkeeping, and there are a few places where Proposition 3 is used.

4 Closure Under Intersection

The Completeness Theorem of the last section can be modified to give a result for subset spaces closed under intersection. For the semantics, write $T \models_i \phi$ to mean the obvious restriction of the $\models$ relation to models closed under intersection.

For the proof theory, we add the instances of axiom scheme

$$(\diamond\Box\phi \ \wedge \ \diamond\Box\psi) \rightarrow \diamond\Box(\phi \wedge \psi) \ .$$

Informally, if a two properties are persistent when we take finer observations, and if they are both possible, then their conjunction is possible. We can take the intersection of any two sets which verify the properties separately.

We write $T \vdash_i \phi$ for the entailment relation generated by the axioms for set spaces and these new axioms.

Example As an example of the use of our formalism, we give an informal proof that the intersection of two open sets is open. Formally, suppose that A and B belong to $\mathcal{A}$. We show

$$A \rightarrow \diamond KA, B \rightarrow \diamond KB \ \vdash_i \ A \wedge B \rightarrow \diamond K(A \wedge B) \ .$$

Assuming A and B, we have $\diamond KA$ and $\diamond KB$. Since A is atomic, $A \rightarrow \Box A$. By a normality axiom, $KA \rightarrow K\Box A$. By the cross axiom, $KA \rightarrow \Box KA$. Similarly, $KB \rightarrow \Box KB$. So $\diamond\Box KA \wedge \diamond\Box KB$. By the axiom of intersection, $\diamond\Box(KA \wedge KB)$. Hence $\diamond(KA \wedge KB)$, and therefore $\diamond K(A \wedge B)$. This completes the proof.

This kind of example is important because it shows that we are getting at the underpinnings of topology. Another way to look at this is to say that we are providing a more formal justification for the discussion of the axioms of topology in terms of observations, given by Vickers [Vi].

Theorem 8 *(Soundness and Completeness For Subset Spaces Closed Under Intersection) For all T and ϕ, $T \models \phi$ iff $T \vdash \phi$.*

The additional result about m-theories relative to $\vdash_i$ is the following:

Proposition 9 *If $T \xrightarrow{\diamond} U_1$ and $T \xrightarrow{\diamond} U_2$, then there is V such that $U_1 \xrightarrow{\diamond} V$ and $U_2 \xrightarrow{\diamond} V$.*

correspond to the "easy" parts of the subject. This correspondence would be more exact if if we could show that this fragment were of a lower complexity than the whole subject.

We also are working on an understanding of continuity in these terms. This would be crucial not only for the topic of the present paper, but also in connection with the knowledge-theoretic analysis of recursive function theory.

In a different direction, Konstantinos Georgatos is considering an enrichment of our framework to add action. The idea is to formalize the reasoning about process observation, and to thereby make a connection between the logical system introduced here and the work on observations and tests in the theoretical computer science literature.

7 References

[BR] Bernstein, A. and Robinson, A. 1966. Solution of an invariant subspace problem of P.R. Halmos and K.T. Smith. *Pacific J. Math.* 16:421-431.

[CM] Chandy, M. and Misra, J. 1986. How processes learn. *Distributed Computing* 1:40-52.

[FHV] Fagin, R., Halpern, J., and Vardi, M. 1988. A Model-Theoretic Analysis of Knowledge. Research Report RJ 6461, IBM.

[Hi] Hintikka, J. 1962. *Knowledge and Belief*, Ithaca: Cornell University Press.

[HM] Halpern, J. and Moses, Y. 1984. Knowledge and common knowledge in a distributed environment, *Proc. 3rd ACM Symposium on Principles of Distributed Computing*, New York: ACM, pp. 50-61.

[Kr] Kreisel, G. 1967. Axiomatizations of nonstandard analysis that are conservative extensions of formal systems for classical standard analysis. *Applications of Model Theory to Algebra, Analysis, and Probability*, W.A.J. Luxemburg (ed.). New York: Holt, Rinehart and Winston, pp. 93-106.

[Ma] Makinson, D. 1966. On some completeness theorems in modal logic. *Zeit. f. Math. Logik* 12:379-384.

[Pa1] Parikh, R. 1967. A conservation result. *Applications of Model Theory to Algebra, Analysis, and Probability*, W.A.J. Luxemburg (ed.). New York: Holt, Rinehart and Winston, pp. 107-108.

[Pa2] Parikh, R. 1984. Logics of knowledge, games and dynamic logic, *FST-TCS, LNCS 181.* Berlin; New York: Springer-Verlag, pp. 202-222.

[PR] Parikh, R. and Ramanujam, R. 1985. Distributed computing and the logic of knowledge. *Logics of Programs, LNCS 193*, R. Parikh (ed.). Berlin; New York: Springer-Verlag, pp. 256-268.

[Sh] Shin, S. J. 1991. Valid Rreasoning and Visual Representation, PhD Thesis, Stanford University.

[Vi] Vickers, S. 1989. *Topology via Logic*, Cambridge: Cambridge University Press.

5 Closure Under Union

Our final completeness result is for subset spaces closed under both intersection and finite union. We add to our list of axioms all of the instances of the following *union scheme*:

$$L\diamond(\phi \wedge K\chi) \ \wedge \ L\diamond(\psi \wedge K\chi) \quad \to \quad \diamond(L\diamond\phi \ \wedge \ L\diamond\psi \ \wedge \ K\diamond\chi) \,.$$

This axiom is quite complicated, and the reader may wish to verify its soundness on spaces closed under unions.

We write $\models_{i,u}$ and $\vdash_{i,u}$ for the obvious semantic and proof theoretic concepts.

Theorem 10 *(Soundness and Completeness For Subset Spaces Closed Under Intersection and Union) For all T and ϕ, $T \models_{i,u} \phi$ iff $T \vdash_{i,u} \phi$.*

The proof of the Completeness Theorem here involves several complications not found in the earlier results. Instead of going into such details, we merely mention the main new result about m-theories.

Definition A m-theory T is a **greatest lower bound (glb)** of two m-theories U and V if

(a) $T \xrightarrow{L\diamond} U$ and $T \xrightarrow{L\diamond} V$

(b) If $T \xrightarrow{L} T'$, then either $T' \xrightarrow{L\diamond} U$ or $T' \xrightarrow{L\diamond} V$.

Lemma 11 *Suppose that $W \xrightarrow{\diamond} U$ and $W \xrightarrow{\diamond} V$. Then there is a glb T of U and V such that $W \xrightarrow{\diamond} T$.*

We conjecture that the Completeness Theorem for subset spaces which are lattices (this section) will extend to the smaller class of topological spaces.

6 Further Areas of Investigation

There are several other matters to investigate as part of an overall project of understanding topology in knowledge-theoretic terms. What we have done here is to take the first step by obtaining a logic for the sets and points. We would like to go on and consider concepts such as compactness. The goal would be to find the appropriate modal language and again prove completeness results.

We would also like to know the complexity of all of these theories. In a sense, we claim that the logics represent a kind of knowledge-theoretic core of topological reasoning, and