Modal Logic S4F and the minimal knowledge paradigm

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1 Introduction

In this paper, we propose a new semantic model and the corresponding syntactic characterization for autoepistemic reasoning. In contrast with the well-known work of Moore [Moo85], who proposed an autoepistemic logic of self-belief, our goal is to propose an autoepistemic logic of self-knowledge.

We are interested in formalizing reasoning of an ideally rational agent with full power of introspection, used by the agent to reflect on his/her knowledge. Initial assumptions of an agent are encoded as a theory in a propositional modal language with a single modality K, which we interpret as *is known*.

In the case when K is interpreted as an operator of *self-belief*, comprehensive analysis of autoepistemic reasoning was given by Moore in [Moo84, Moo85]. Some of his analysis carries over to the case of reasoning about self-knowledge. For example, theories describing the agent's total knowledge must be stable (for the definition see Section 3, or [Moo85]). Moreover, reasoning about self-knowledge is nonmonotonic, by the very nature of autoepistemic reasoning (see [Moo85, Lev90]).

The key concept behind our approach to autoepistemic logic of self-knowledge is the paradigm of minimal knowledge, extensively studied in the past [HM85b, Sho87, Lev90, LS90, Lif91]. Usually, knowledge sets are represented as theories associated with collections of possible worlds (or, equivalently, Kripke **S5**-models with a universal accessibility relation; such models will be referred to in this paper as *universal S5-models*). In order to minimize knowledge, the collection of possible worlds is maximized. Up to now, the standard way to minimize knowledge has been to maximize collections of possible worlds with respect to the inclusion relation. We propose an essential modification to this approach. Namely, we recognize that the state of a possible world is determined not only by the objective facts true in the world, but also by what is *known* in this world. Consequently, we allow only such extensions of a collection of possible worlds that preserve their states, understood as explained above. We show that our logic avoids some counterintuitive properties of earlier proposals while retaining much of their expressive power.

The formalism we obtain is nonmonotonic, simply because the reasoning it models is nonmonotonic. We study the relation of our logic to other nonmonotonic formalisms and discover interesting connections with the nonmonotonic logics of McDermott and Doyle [MD80, McD82].

As a byproduct, we obtain a syntactic characterization of our system. Relations to default and autoepistemic logics are also discussed.

2 Minimal-knowledge paradigm: a short overview

The first attempt to revise Moore's autoepistemic logic so that it would capture the notion of knowledge rather than belief is due to Halpern and Moses [HM85b]. Their approach is based on the idea of minimizing knowledge (they use the term "*only knowing*").

An *HM-knowledge model* (or, simply, an *HM-model*) is a set M of propositional interpretations. Each interpretation identifies a possible world or a possible state of the world and defines which objective (that is, modal-free) statements are true in the world. A sentence $K\psi$, *ψ is known*, is true in a world $\alpha \in M$, if for each $\beta \in M$, ψ is true in β. Thus, the agent *knows* a sentence if it is true in all possible worlds. The *theory* (or the *knowledge set*) of an HM-model M, denoted $Th(M)$, is defined as the set of all modal formulas true in every possible world from M.

Clearly, HM-knowledge models are simply universal S5-models for the well-known modal logic S5. Monotonic logic S5 seems to be adequate for modeling reasoning about knowledge (see [HM85b]), but not if the agent is reflecting on what is known and not known to him/her. In the case when the agent possesses introspection powers, in the absence of any initial assumptions the agent should conclude $\neg Kp$ (p is unknown) for each atomic sentence p. But logic S5, as well as all other monotonic modal logics for that matter, does not allow such derivations.

Instead of considering all HM-models in which agent's initial assumptions I are satisfied, Halpern and Moses [HM85b] propose to restrict attention to, as they call them, *minimal* models of I. An HM-knowledge model M of I is *minimal* if all the formulas of I are true in M, and for no proper superset N of M, I is true in N. In other words, an HM-model M of I is *minimal* if it is an *inclusion-maximal* HM-model of I. If I has a unique minimal HM-model, then I is said to be *HM-honest*, and the formulas true in this model are regarded as logical (auto)epistemic consequences of I.

Clearly, if M' and M'' are two HM-knowledge models and $M' \subseteq M''$, then the set of all objective formulas true in M' contains the set of all objective formulas true in M''. That is, maximizing HM-models is equivalent to minimizing (objective) knowledge. However, minimizing knowledge not by directly comparing knowledge sets but by comparing the corresponding models has important advantages. Instead of dealing with knowledge sets that are infinite and difficult to explicitly describe, models, which often admit finite representations (for example, when the set of initial assumptions I is finite), are compared and the same effect is obtained.

The approach of [HM85b] was further extended by Lin and Shoham [LS90] to handle two epistemic notions within one system: knowledge and assumption. Their logic GK (logic of ground knowledge) contains two modalities: K and A ("is assumed"). In [LS90], there are two important ideas concerning minimizing knowledge: (1) knowledge should be minimized with respect to a fixed assumption set, and (2) in the preferred (maximal) model, the knowledge set of the model should coincide with the assumption set of the model. In this paper, we show how to achieve both desiderata in a unimodal logic.

Building on the ideas of Lin and Shoham, Lifschitz [Lif91] proposed a modification to the logic GK and extended his system to the predicate case.

The logic of [HM85a] has the following logically counterintuitive property: it is not conservative with respect to introducing explicit definitions. That is, if F is a formula of the language, adding axiom $q \equiv F$ to a theory I, where q is a new propositional letter, may affect theorems not containing q, and may even convert an HM-honest theory into a dishonest one. (Example: consider the empty theory and its extension by $q \equiv Kp$.) This property is inherited by logics based on the same idea of minimizing knowledge by maximizing collections of possible worlds with respect to inclusion [Lif91, LS90]. In the case of the logic GK [LS90], by an explicit definition we understand both $q \equiv F$ and $K(q \equiv F)$, the latter formula stating that the agent *knows* about the new definition.

All methods of minimizing knowledge discussed in [HM85a, LS90, Lif91], as well as our approach introduced below, can be regarded as special cases of a very general method proposed by Shoham [Sho87], which yields different notions of minimality by allowing us to vary the class of models and the preference relation between them.

In the next section we use the ideas of [Sho87] and [LS90] as a starting point for our proposal for a logic of minimal knowledge. We will argue that our logic is simpler (it is a single-modality logic) than bimodal systems of [LS90, Lif91], yet, for an important class of theories without nested modalities, equally expressive. In the same time, it is conservative with respect to additions of new definitions, a property that other logics do not have.

3 Minimizing knowledge differently

The approaches of [HM85b, Lif91] to the problem of minimizing knowledge identify a possible state of the world with a propositional interpretation, that is, with the set of objective sentences valid in the world. But it is well known that, "in reality", a state of the world depends not only on the objective truths in the world but also on what agents (agent) know(s) in this world (see [Par89, HM90], where this observation is exploited for the many-knower situation).

Let us discuss first the method of extending knowledge models mentioned in the introduction. Consider two HM-knowledge models M and N (recall that a knowledge model is just a set of propositional valuations). Suppose that M is a proper subset of N. Is it indeed justifiable to say that the model N is an extension of the model M? Take a world $\alpha \in M$. Clearly, the truth of a formula $K\psi$ in α depends on whether α is treated as a world of M or N and, in general, $K\psi$ may be true in α treated as a world of M and may not be true when α is treated as a world of N. Therefore, world α considered as a world of M is different from world α when considered as a world in N. Consequently, when we minimize knowledge by means of simple inclusion relation, the same propositional interpretation may represent different states of the world, depending on whether it is considered as a member of M or N. But if so, N should not be considered an extension of M.

Before we present our approach, let us introduce some notation. We will consider a fixed propositional modal language $\mathcal{L}$ with the only modality K interpreted as "is known". We will use also the dual modality $M = \neg K \neg$ meaning "is possible" (if the negation of ψ is not known to an agent, then the agent considers ψ possible). The language of objective formulas will be denoted $\mathcal{L}_0$. Throughout the paper, I always denotes the theory encoding the agent's initial assumptions.

What is the correct notion of extension of a model? What is the total knowledge set that might be associated with I? Following Halpern and Moses [HM85b] and Moore [Moo84, Moo85], we will require that knowledge sets be stable.

Formally, a theory $T \subseteq \mathcal{L}$ is *stable* if T is closed under propositional consequence, necessitation and if for every $\varphi \in \mathcal{L}$, if $\varphi \notin T$ then $\neg K\varphi \in \mathcal{L}$. Two properties of stable theories are crucial for our discussion. First, a theory T is stable if and only if it is the theory of an S5-model with a universal accessibility relation [Moo84]. In particular, theories of HM-models (knowledge sets) are stable. Secondly, for every set S of modal-free formulas there is a *unique* stable theory T such that $T \cap \mathcal{L}_0 = Cn(S)$. (Here, Cn denotes the operator of propositional consequence in the language $\mathcal{L}_0$.) This unique stable theory will be denoted in the paper by $ST(S)$.

The main novelty of our approach is the way in which we test whether a model is maximal. In order to precisely describe our approach, we will need somewhat more complicated Kripke models than just S5-models (HM-models).

Definition 3.1 By a *knowledge model* (we use the same term as before; we hope that no ambiguity will arise) we mean a Kripke model of the form $\mathcal{M} = \langle M_1, M_2, R, V \rangle$, where $M = M_1 \cup M_2$ is the set of possible worlds, M_1 and M_2 are disjoint, M_2 is nonempty, for each $\alpha \in M$, $V(\alpha)$ is a propositional valuation and R is the accessibility relation on $M = M_1 \cup M_2$ such that $\alpha R \beta$ if and only if $\beta \in M_2$ or $\alpha \in M_1$.

That is, in a knowledge model $\langle M_1, M_2, R, V \rangle$ the worlds in M_2 are accessible from all other worlds, and the worlds in M_1 are accessible only from the worlds in M_1. As R is uniquely defined by M_1 and M_2, we will identify a knowledge model $\langle M_1, M_2, R, V \rangle$ with a triple $\langle M_1, M_2, V \rangle$.

Every S5-model $\langle M, V \rangle$, with a universal accessibility relation, can be identified with a knowledge model $\langle \emptyset, M, V \rangle$. Thus, slightly abusing notation, we may say that the class of knowledge models extends the class of universal S5-models (that is, HM-models).

The relation $\langle \mathcal{M}, \alpha \rangle \models \psi$ is defined in a standard Kripke-semantics way. If $\alpha \in M_1$, then $\langle \mathcal{M}, \alpha \rangle \models K\psi$ if ψ is true in all worlds, and if $\alpha \in M_2$, then $K\psi$ is true in α if ψ is true in all worlds in M_2. This observation suggests the following intuition: the worlds in M_2 are those where the agent knows more facts, and the worlds in M_1 are those where the agent knows fewer.

Now we will introduce the preference relation between knowledge models or, equivalently, describe a way of extending a knowledge model.

Definition 3.2 Let $\mathcal{N} = \langle N, M, V \rangle$ be a knowledge model. Consider the universal S5-model $\mathcal{M} = \langle M, V|M \rangle$ (that is, the "upper cluster" of $\mathcal{N}$). We will say that $\mathcal{N}$ *is preferred over* $\mathcal{M}$, or that $\mathcal{N}$ *extends* $\mathcal{M}$ if for some formula $\psi \in \mathcal{L}_0$, $\mathcal{M} \models \psi$, but for some $\alpha \in N$, $\langle \mathcal{N}, \alpha \rangle \not\models \psi$.

As we have said earlier, $\mathcal{M}$ is a special case of a knowledge model with an empty lower cluster; compare with [LS90], where in a preferred model the knowledge set coincides with the assumption set, but the preference relation involves also models without this property. If $\mathcal{N}$ is preferred over an S5-model $\mathcal{M}$, then $\mathcal{N}$ contains an isomorphic copy of $\mathcal{M}$. In addition, all the worlds in M have the same meaning (state) no matter whether treated as worlds of $\mathcal{M}$ or $\mathcal{N}$, that is, for each modal formula, its truth value in each $\alpha \in M$ is the same in $\mathcal{M}$ as in $\mathcal{N}$. In other words, $\mathcal{N}$ is a true extension of $\mathcal{M}$.

Definition 3.3 Let $\mathcal{M} = \langle M, V \rangle$ be an S5-model. We say that $\mathcal{M}$ is a *minimal-knowledge model of* I, if $\mathcal{M} \models I$, and for every knowledge model $\mathcal{N}$ such that $\mathcal{N}$ is preferred over $\mathcal{M}$, $\mathcal{N} \not\models I$. If there is exactly one minimal-knowledge model $\mathcal{M}$ of I, then I is said to be *honest*, and its theory $\{\psi : \mathcal{M} \models \psi\}$ is the set of autoepistemic consequences of I.

Intuitively, we can say that $\mathcal{M}$ is a minimal-knowledge model of I if its knowledge set will not decrease even if we add worlds with less information, as long as I remains valid and the states of all worlds in $\mathcal{M}$ are preserved after extension.

Example 1. $I_1 = \{Kp\}$. There is exactly one minimal-knowledge model, which coincides with the (only) HM-minimal model. This theory has no stable expansions, which agrees with the interpretation of K as "is believed" in Moore's logic.
Example 2 $I_2 = \{Kp \supset p\}$. There is again a unique minimal-knowledge model (consisting of all propositional interpretations). Likewise, there is a unique HM-minimal model. This theory has two stable expansions. One of them corresponds to the minimal knowledge model (HM-minimal model), and another contains the formula p.
Example 3. $I_3 = \{\neg Kp \supset q\}$. This theory is dishonest according to [HM85b]: it has two HM-minimal models. One model consists of all interpretations in which q is true, and the other consists of all interpretations in which p is true. The latter one is counterintuitive. In our case, this latter model is not a minimal-knowledge model. So I_3 is honest and q is its autoepistemic consequence, in agreement with intuition.

4 Minimal-knowledge logic as nonmonotonic logic S4F

McDermott and Doyle [MD80, McD82] proposed a general scheme of defining expansions, that is possible sets of knowledge an agent is justified in assuming by reasoning introspectively from initial beliefs. Let $\mathcal{S}$ be a modal logic. A consistent set of formulas T is an *$\mathcal{S}$-expansion* for a set of initial knowledge I if

$$T = Cn_{\mathcal{S}}(I \cup \{\neg K\psi : \psi \notin T\}),$$

where by $Cn_{\mathcal{S}}$ we denote the consequence operator in a modal logic $\mathcal{S}$ (in the language $\mathcal{L}$). The epistemic consequence operator is defined as the intersection of all $\mathcal{S}$-expansions for I (alternatively, we can define I to be *$\mathcal{S}$-honest*, if there is exactly one $\mathcal{S}$-expansion of I and regard only $\mathcal{S}$-honest theories as consistent ones). The corresponding nonmonotonic logic is referred to as the *nonmonotonic modal logic $\mathcal{S}$*. It is well-known that $\mathcal{S}$-expansions are stable. In particular, for each $\mathcal{S}$-expansion T there is a universal S5-model $\mathcal{M}$ such that $T = Th(\mathcal{M})$.

The McDermott-Doyle approach has proved to be very powerful [Shv90, MST91, Tru91]. It admits constructive syntactic characterization [Shv90, MST91] and natural preferred-model semantics [Sch91]. It contains Moore's autoepistemic logic as a special case (when $\mathcal{S}$ = KD45) [Shv90], and Reiter's default logic can be embedded into nonmonotonic $\mathcal{S}$ for some suitable logics $\mathcal{S}$ [Tru91]. The great advantage of this approach is that a whole family of epistemic formalisms can be investigated in a uniform way using techniques developed in the area of (classical) modal logic.

The logic of minimal knowledge proposed in the previous section also falls into the McDermott-Doyle scheme. The class of knowledge models defined above is well-known in classical modal logic and was studied extensively by Segerberg [Seg71]. The modal logic characterized by this class of models is known as the modal logic S4F, and its axiomatization consists of the schemata of the logic S4 plus the axiom schema $M\varphi \wedge MK\psi \supset K(M\varphi \vee \psi)$. The following theorem is a special case of a more general result of [Sch91].

Theorem 4.1 *An* S5*-model $\mathcal{N}$ is a minimal-knowledge model for I if and only if the theory of $\mathcal{N}$ is an* S4F*-expansion of I.* □

This result yields a proof-theoretic characterization of the logic of minimal knowledge in terms of the consequence operator in the logic S4F and allows us to apply techniques developed in [Shv90, MST91, Sch91] to investigate it.

Another consequence of Theorem 4.1 is that minimal-knowledge models are preserved when new definitions are introduced. Let q be a propositional variable distinct from all variables in a propositional language $\mathcal{L}_0$. By $\mathcal{L}_0^q$ we denote the extension of $\mathcal{L}_0$ by q, and by $\mathcal{L}^q$ the modal language corresponding to $\mathcal{L}_0^q$. We say that a theory $U \subseteq \mathcal{L}^q$ is a *conservative* extension of a theory $T \subseteq \mathcal{L}$ if $T = U \cap \mathcal{L}$.

Consider a modal logic $\mathcal{S}$. Let $Cn_{\mathcal{S}}$ denote, as before, the consequence operator in $\mathcal{S}$ in the language $\mathcal{L}$, and let $Cn_{\mathcal{S}}^q$ be the consequence operator in $\mathcal{S}$ in the language $\mathcal{L}^q$. Consider a formula $\eta \in \mathcal{L}$. For a theory $I \subseteq \mathcal{L}$, define

$$I^{q,\eta} = I \cup \{q \equiv \eta\}.$$

The rule of uniform substitution implies that

$$Cn_{\mathcal{S}}(I) = Cn_{\mathcal{S}}^q(I^{q,\eta}) \cap \mathcal{L},$$

that is, $Cn_{\mathcal{S}}^q(I^{q,\eta})$ is a conservative extension of $Cn_{\mathcal{S}}(I)$.

The next theorem shows that there is a natural *one-to-one* and *onto* mapping between $\mathcal{S}$-expansions for I and $I^{q,\eta}$. Moreover, $\mathcal{S}$-expansions for $I^{q,\eta}$ are conservative extensions for corresponding $\mathcal{S}$-expansions for I.

Theorem 4.2 *Let $\mathcal{S}$ be any normal modal logic. Let $I \subseteq \mathcal{L}$. Then*

(i) *A stable theory T is an $\mathcal{S}$-expansion for I if and only if $Cn_{\mathcal{S}}^q(T \cup \{\eta \equiv q\})$ is an $\mathcal{S}$-expansion of $I^{q,\eta}$;*

(ii) *each $\mathcal{S}$-expansion of $I^{q,\eta}$ is of the form $Cn_{\mathcal{S}}^q(T \cup \{\eta \equiv q\})$ for some stable theory $T \subseteq \mathcal{L}$.*

There is a simple test for "groundedness" of an epistemic formalism for reasoning about knowledge. A formula of the form

$$K\varphi_1 \wedge \ldots \wedge K\varphi_n \supset \psi,$$

where $\varphi_1, \ldots, \varphi_n$ and ψ are objective, $n \geq 0$, is called a *positively determined clause.* Theories consisting of positively determined clauses should be honest in any reasonable logic of (minimal) knowledge [Kon88]. The autoepistemic logic of Moore (nonmonotonic KD45) does not pass this test (not surprisingly perhaps, as it is a logic of self-belief and not self-knowledge) and the same is true for the nonmonotonic logic S5. Our next theorem shows that, although the logic S4F is very strong ("almost" S5), in its nonmonotonic variant and, consequently, in our logic of minimal knowledge, every theory I is honest.

Theorem 4.3 *If I consists of positively determined clauses, then I has a unique minimal-knowledge model (the canonical* S5*-model of $Cn_{S5}(I) \cap \mathcal{L}_0$) or, equivalently, a unique* S4F*-expansion.*

In fact, it is easy to show that even for a very weak modal logic N, which does not contain *any* modal axiom schemata, if a theory consists of positively determined clauses, then it possesses an N-expansion [MT90]. In addition, it is also well known that for every two modal logics $\mathcal{S}$ and $\mathcal{T}$ contained in S5 and such that $\mathcal{S} \subseteq \mathcal{T}$, if T is an $\mathcal{S}$-expansion for a theory I then T is a $\mathcal{T}$-expansion for I. Consequently, it follows that the assertion of Theorem 4.3 holds for every modal logic containing N and contained in S4F. But it turns out that the logic S4F is maximal in this respect.

Theorem 4.4 *If $\mathcal{S}$ is a modal logic* properly containing S4F *and contained in* S5 *then there is a theory I consisting of positively determined clauses such that $\mathcal{S}$ has more than one $\mathcal{S}$-expansion.*

5 Expressive power of logic of minimal knowledge (nonmonotonic logic S4F)

In this section we will show that the logic of minimal knowledge (nonmonotonic logic S4F) naturally generalizes default logic and autoepistemic logic.

Autoepistemic logic and default logic are two essentially different formalisms. Our results show that both are embeddable into the same logic. We explore implications of this result here. By comparing the corresponding embeddings we gain insight into the nature of the differences between default and autoepistemic logics.

A *default* is an expression of the form

$$\frac{\varphi, M\psi_1, \ldots, M\psi_n}{\eta}. \tag{1}$$

where $\varphi, \psi_1, \ldots, \psi_n, \eta \in \mathcal{L}_0$. For more details on default logic the reader is referred to [Rei80]. If we encode a default d given by (1) as

$$emb(d) = K\varphi \wedge KM\psi_1 \wedge \ldots \wedge KM\psi_n \supset K\eta$$

and, for a default theory (D, W), define

$$emb(D, W) = \{K\varphi : \varphi \in W\} \cup \{emb(d) : d \in D\},$$

then we have the following result [Tru91] establishing embeddibility of default logic into nonmonotonic logic S4F.

Theorem 5.1 *Let (D, W) be a default theory and let $S \subseteq \mathcal{L}_0$ be consistent and closed under propositional consequence. Theory S is an extension of (D, W) if and only if $ST(S)$ is an* S4F*-expansion of $emb(D, W)$.* □

In theorem 5.1, logic S4F can be replaced by any modal logic (not necessarily normal) contained in S4F and satisfying axiom T [Tru91]. But it turns out that S4F is a maximal logic contained in S5 and sound with respect to the translation *emb*.

Theorem 5.2 *Let S be a modal logic* properly *containing* S4F *and contained in* S5*. There is a default theory (D, W) such that (D, W) has exactly one extension and $emb(D, W)$ has at least two S-expansions.*

We will show now how Theorems 4.1 and 5.1 can be used to establish a minimal-knowledge model semantics for default logic. If S is a consistent set of propositional sentences, then the *canonical model for S* is the universal S5-model $\mathcal{M}$ whose worlds are all the propositional interpretations in which S is true. It is well known that if $\mathcal{M}$ is the canonical model for S, then $Th(\mathcal{M}) = \{\varphi : \mathcal{M} \models \varphi\} = ST(S)$ ([Moo85]). Theorems 4.1 and 5.1 imply the following corollary.

Corollary 5.3 *Let (D, W) be a default theory and let $S \subseteq \mathcal{L}_0$ be consistent and closed under propositional consequence. Theory S is an extension of (D, W) if and only if the canonical* S5*-model for S is a minimal-knowledge model of $emb(D, W)$.* □

Corollary 5.3 makes it possible to adapt the minimal-knowledge semantics for default logic. The obtained semantics is closely related to the semantics proposed in [GC90].

Definition 5.1 A *default model* is a pair $\mathcal{V} = \langle V_1, V_2 \rangle$, where V_1 and V_2 are sets of propositional valuations of $\mathcal{L}$ and $V_2 \neq \emptyset$. A default d of the form (1) is *valid* in a default model $\mathcal{V} = \langle V_1, V_2 \rangle$ if the following conditions hold:

(1) whenever φ is true in all valuations of $V_1 \cup V_2$ and each ψ_i, $1 \leq i \leq n$, is true in at least one valuation of V_2, then η is true in all valuations of $V_1 \cup V_2$;

(2) whenever φ is true in all valuations of V_2 and each ψ_i, $1 \leq i \leq n$, is true in at least one valuation of V_2, then η is true in all valuations of V_2.

We write $\mathcal{V} \models d$ to denote that d is valid in a default model $\mathcal{V}$.

We say that a formula $\varphi \in \mathcal{L}_0$ is valid in a default model $\mathcal{V} = \langle V_1, V_2 \rangle$ if φ is valid in every valuation from $V_1 \cup V_2$. We write $\mathcal{V} \models \varphi$ to denote that φ is valid in $\mathcal{V}$.

We say that a default theory (D, W) is *valid* in a default model $\mathcal{V}$ if all defaults in D and all formulas in W are valid in $\mathcal{V}$. We write $\mathcal{V} \models (D, W)$ if (D, W) is valid in $\mathcal{V}$.

There is an obvious analogy between the definitions of knowledge models and default models. Further extending this analogy, we define next the notion of a minimal-knowledge default model.

Definition 5.2 Let (D, W) be a default theory. A default model $\langle \emptyset, V \rangle$ is a *minimal-knowledge model* for (D, W) if

(MAX1) $\langle \emptyset, V \rangle \models (D, W)$; and

(MAX2) for every set of valuations V', if $\langle V', V \rangle \models (D, W)$ then $V' \subseteq V$.

The following theorem is an immediate corollary to Theorem 5.1 and Corollary 5.3.

Theorem 5.4 *Let (D, W) be a default theory and let $S \subseteq \mathcal{L}$ be consistent. Theory S is an extension of (D, W) if and only if $S = Th(\mathcal{V})$, where $\mathcal{V}$ is a minimal-knowledge model for (D, W).* □

We will compare our semantic characterization of default logic with that of Guerreiro and Casanova [GC90]. Let us recall basic elements of their approach. Given a set V of valuations, we define $\Sigma(V)$ to be the largest set V' of valuations such that for any default of the form (1), if $\varphi \in Th(V')$ and $\neg\psi_i \notin Th(V)$, $1 \leq i \leq n$, then $\eta \in Th(V')$. It is easy to show that for each V, $\Sigma(V)$ is well-defined. Guerreiro and Casanova [GC90] proved the following theorem.

Theorem 5.5 *Let (D, W) be a default theory and let $S \subseteq \mathcal{L}_0$ be consistent. Theory S is an extension of (D, W) if and only if $S = Th(V)$, for some set V of valuations such that $V = \Sigma(V)$.* □

By a *strong default model* we mean a default model $\langle V_1, V_2 \rangle$ with $V_1 \cap V_2 = \emptyset$. A default model $\mathcal{V} = \langle \emptyset, V \rangle$ is a *weak minimal-knowledge* model of (D, W) if

(MAX1′) $\langle \emptyset, V \rangle \models (D, W)$; and

(MAX2″) for every strong default model $\langle V', V \rangle$ such that $\langle V', V \rangle \models (D, W)$, $V' = \emptyset$.

It follows directly from the definitions that each minimal-knowledge model for a default theory (D, W) is a weak minimal-knowledge model as well. It is also not hard to prove that each weak minimal-knowledge model for (D, W) is a minimal-knowledge model for (D, W).

Clearly, a set of valuations V can be identified with a default model $\langle \emptyset, V \rangle$ and the operator Σ can be treated as defined on default models of the form $\langle \emptyset, V \rangle$. It is easy to see that for a default model $\mathcal{V}$ such that $\mathcal{V} \models (D, W)$, $\mathcal{V} = \Sigma(\mathcal{V})$ if and only if $\mathcal{V}$ is a weak minimal-knowledge model for (D, W) and, by the remarks in the preceding paragraph, if and only if $\mathcal{V}$ is a minimal-knowledge model for (D, W). Thus, the minimal-knowledge model semantics described here and the semantics of Guerreiro and Casanova are equivalent.

Remark 5.1 When we define a knowledge model $\langle M_1, M_2, V \rangle$ (in Section 3) we could, in principle, require that for each $\alpha \in M_1$, $\beta \in M_2$, $V(\alpha)$ be different from $V(\beta)$ (which would yield a notion of a model analogous to a strong default model). We have just seen that such a restriction on the class of models has no effect when defining minimal-knowledge default models. Similarly, for modal translations of default theories, restricting the class of models used in defining minimal-knowledge models will have no effect. The same class of minimal-knowledge models will be obtained. In general it is essential to allow the possibility that some worlds in M_1 and M_2 agree on all propositional variables (see [Sch91] for examples). Forbidding such a possibility, in particular, yields logics of knowledge that do not preserve minimal-knowledge models under the introduction of new definitions $q \equiv \eta$.

Somewhat surprisingly, autoepistemic logic can also be embedded into the logic of minimal knowledge. Thus, the logic of minimal knowledge (nonmonotonic S4F) is a common generalization of default and autoepistemic logics, which makes formal comparisons of these logics possible.

For each formula $\varphi \in \mathcal{L}$, by $\hat{\varphi}$ we denote the formula obtained from φ by replacing each ocurrence of K in φ by MK. For a theory $I \subseteq \mathcal{L}$, we define $\hat{I} = \{\hat{\varphi} : \varphi \in I\}$. The following theorem was proved in [Shv91] for modal logic SW5 instead of S4F. However, the proof remains valid if we replace SW5 by S4F everywhere.

Theorem 5.6 *Let $T \subseteq \mathcal{L}$ be consistent. Then T is a stable (*KD45*-) expansion for I if and only if T is an* S4F*-expansion of $\hat{I}$ or, equivalently, the theory of a minimal-knowledge model of $\hat{I}$.* □

The difference between default logic and autoepistemic logic is now evident. An autoepistemic formula *if φ is believed then η*, expressed in autoepistemic logic as $K\varphi \supset \eta$, is encoded in nonmonotonic S4F as $MK\varphi \supset \eta$, while default *if φ is established, then establish η* ($\frac{\varphi}{\eta}$) is expressed as $K\varphi \supset \eta$. In other words, default logic is a logic of self-knowledge, since the meaning of the modalities implicitly present in default logic coincides with the meaning of K and M in nonmonotonic S4F (the logic of minimal knowledge). On the other hand, autoepistemic logic is a logic of self-belief, since the meaning of the modality K of the autoepistemic logic is the same as that of MK (possibly known) in the logic S4F.

6 Comparison with other minimal-knowledge logics

Let us reformulate basic notions of [LS90]. By an *LS-structure* we mean a quadruple $\mathcal{M} = \langle w, M_K, M_A, V \rangle$, where w is a world, M_K and M_A are sets of worlds, and for each $\alpha \in \{w\} \cup$

$M_K \cup M_A$, V_α is a propositional valuation. The satisfaction relation $\mathcal{M} \models \varphi$ between LS-structures and bimodal propositional formulas is defined as follows:

$\langle w, M_K, M_A, V\rangle \models p$, where p is a propositional variable, if $V(w)(p) = 1$,

The Boolean connectives are treated in a standard way and the modalities are handled by:

$\langle w, M_K, M_A, V\rangle \models K\psi$ if for each $\alpha \in M_K$, $\langle \alpha, M_K, M_A, V\rangle \models \psi$,

$\langle w, M_K, M_A, V\rangle \models A\psi$ if for each $\alpha \in M_A$, $\langle \alpha, M_K, M_A, V\rangle \models \psi$.

If $\mathcal{M} = \langle w, M_K, M_A, V\rangle$ and $\mathcal{N} = \langle v, N_K, N_A, W\rangle$ are two LS-structures, then we say that $\mathcal{M}$ is *LS-preferred* over $\mathcal{N}$, if

(1) $M_A = N_A$ and $M_K \supset N_K$;

(2) V coincides with W on $N_K \cup N_A$;

(3) there exists $\alpha \in M_K \setminus N_K$ and an objective formula ψ such that for every $\beta \in N_K$, $V(\alpha)(\psi) \neq V(\beta)(\psi)$.

An LS-structure $\mathcal{M}$ is an LS-minimal model of a theory I if $\mathcal{M} \models I$, $M_K = M_A$ and for every LS-structure $\mathcal{N}$ such that $\mathcal{N}$ is LS-preferred over $\mathcal{M}$, $\mathcal{N} \not\models I$.

From the definition of LS-minimal models it is clear that, without the loss of generality, we can restrict ourselves to LS-structures of the form $\langle w, M_K, M_A, W\rangle$, where $M_A \subsetneq M_K$. With each such LS-structure $\mathcal{M}$ we can associate the S4F-model $\mathcal{M}^* = \langle M_K \setminus M_A, M_A, W\rangle$. Worlds in $M_K \setminus M_A$ are accessible from the worlds in M_A in the LS-structure $\mathcal{M}$ but are inaccessible in $\mathcal{M}^*$.

Let ψ be an objective formula. Clearly, $\mathcal{M} \models K\psi$ if and only if $\mathcal{M}^* \models K\psi$. In addition, $\mathcal{M} \models A\psi$ if and only if $\mathcal{M}^* \models MK\psi$. Thus the modality A ("is assumed") may be identified with the modality MK (or, equivalently $\neg K \neg K$), "possibly known", and thus, it is closely related with the modality K in the autoepistemic logic (see our interpretation of autoepistemic logic in minimal-knowledge logic and the results in [LS90]).

Let ψ be any formula of Lin-Shoham's bimodal language. By ψ^l we denote the result of substituting $MK (= \neg K \neg K)$ for each occurrence of A in ψ. The formula ψ is called *fully-modalized* if each occurrence of each propositional atom in ψ lies within the scope of a modal operator (A or K). The following proposition follows from the observations made above.

Proposition 6.1 *Let I be a set of fully-modalized formulas without nested modalities of the bimodal language of the logic* GK. *Then $\langle w, M_A, M_A, V\rangle$ is an LS-minimal model of I if and only if $\langle M_A, W\rangle$ is a minimal-knowledge model (as defined in Section 4) of I^l.* □

Results of [LS90] which establish the correspondence between GK and default logic, and between GK and autoepistemic logic of Moore, interpret defaults and autoepistemic formulas as subjective formulas without nested modalities, so for such theories the above theorem applies and the bimodal logic GK can be replaced by our simpler, unimodal logic of minimal knowledge.

7 Conclusions

We propose a new minimal-knowledge model reflecting the idea that a state of the world is determined not only by the objective sentences true in it, but also by agent's knowledge in the state. The resulting logic has intuitive preference semantics and an elegant syntactical characterization as nonmonotonic logic S4F.

Earlier formalisms (Halpern-Moses, Lin and Shoham, Lifschitz) share the unpleasant property of not preserving minimal models with respect to additions of new definitions. Our logic of minimal knowledge (nonmonotonic S4F) preserves minimal models (S4F-expansions) when new denotations are introduced, which is a simple consequence of the corresponding result for normal monotonic modal logics.

Our logic, being unimodal, is simpler than bimodal logic GK of [LS90] and its extension described in [Lif91]. Yet it is applicable in all situations discussed in [LS90, Lif91] if the modality A is replaced by MK. In particular, default logic and autoepistemic logic can be embedded into our system.

Lifschitz's [Lif91] modification of the logic GK has the important advantage that it allows a natural generalization to the predicate case. Our logic of minimal knowledge allows such generalization as well. Moreover, because our knowledge models have two clusters, we can vary domains, which was impossible in the approach of Lifschitz.

Finally, since our logic of minimal knowledge coincides with nonmonotonic S4F, techniques developed to study nonmonotonic logics $\mathcal{S}$ ([McD82, Shv90, MST91, Tru91]) can be applied to our logic. In particular, there are algorithms for computing theories of minimal knowledge models of finite theories (S4F-expansions).

In the framework of McDermott-Doyle, many nonmonotonic modal logics share good properties of S4F, discussed in this paper. We showed that S4F is a *maximal* monotonic modal logic such that these properties hold. These maximality results provide, in addition to intuitively acceptable, clear and natural semantics of nonmonotonic S4F as a logic of minimal knowledge, a strong argument in favor of the importance of this long-overlooked logic for knowledge representation applications.

8 Proofs of the results

Proof of Theorem 4.2: First observe that because $\mathcal{S}$ is normal, we have

$$\eta \equiv q \vdash_{\mathcal{S}} \zeta \equiv \zeta(q/\eta)$$

for each $\zeta \in \mathcal{L}^q$, in particular, for ζ of the form $\neg K\psi$ (here, $\zeta(q/\eta)$ denotes the result of replacing in ζ simultaneously all occurrences of q by η, $\vdash_{\mathcal{S}}$ denotes the provability in $\mathcal{S}$ by using modus ponens and necessitation rules). Since $\mathcal{S}$-expansions are closed under deduction in $\mathcal{S}$, it follows that a theory $S \subseteq \mathcal{L}^q$ is an $\mathcal{S}$-expansion for $I^{q,\eta}$ if and only if

$$S = Cn^q_{\mathcal{S}}(I \cup \{\eta \equiv q\} \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus S\}). \tag{2}$$

(Notice that only the formulas of the form $\neg K\psi$, where $\psi \in \mathcal{L} \setminus S$ are needed on the right hand side of the equation.)

In order to prove the part (i) of the theorem, assume first that T is an $\mathcal{S}$-expansion of I. Then $T \subseteq Cn^q_{\mathcal{S}}(I \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus T\})$. Define $S = Cn^q_{\mathcal{S}}(T \cup \{\eta \equiv q\})$. The theory S is a conservative extension of T with respect to $\mathcal{L}$. Hence, $\mathcal{L} \setminus S = \mathcal{L} \setminus T$. Consequently,

$$S \subseteq Cn^q_{\mathcal{S}}(I \cup \{\eta \equiv q\} \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus S\}).$$

Next, consider a formula $\psi \in \mathcal{L} \setminus S$. Since $\mathcal{L} \setminus S = \mathcal{L} \setminus T$ and by stability of T, $\neg K\psi \in T$. Then, by the definition of S, $\neg K\psi \in S$ and the converse inclusion follows.

Conversely, assume that $S = Cn^q_{\mathcal{S}}(T \cup \{\eta \equiv q\})$, where T is stable, is an $\mathcal{S}$-expansion for $I^{q,\eta}$. Then, (2) holds for S. Since $I \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus S\} \subseteq \mathcal{L}$, and since $T = S \cap \mathcal{L}$,

$$T = S \cap \mathcal{L} = Cn^q_{\mathcal{S}}(I \cup \{\eta \equiv q\} \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus S\}) \cap \mathcal{L} = Cn_{\mathcal{S}}(I \cup \{\neg K\psi : \psi \in \mathcal{L} \setminus T\}).$$

Thus, T is an $\mathcal{S}$-expansion for I.

In order to prove (ii), assume that S is an $\mathcal{S}$-expansion of $I^{q,\eta}$. Then S is stable and, consequently, $T = S \cap \mathcal{L}$ is stable. From (2) it follows that

$$S = Cn^q_{\mathcal{S}}(T \cup \{\eta \equiv q\}).$$

Thus, (ii) follows. □

Proof of Theorem 4.3: The theorem was proved in [MT90] for a modal logic N which is contained in S4F. Hence, I has at least one S4F-expansion.

Assume T_i, $i = 1, 2$, are S4F-expansions of I. Let $\mathcal{N}_i = \langle N_i, V_i \rangle$, $i = 1, 2$, be universal S5-models such that $T_i = Th(\mathcal{N}_i)$, $i = 1, 2$. Clearly, $\mathcal{N}_i \models I$, $i = 1, 2$. Consider the knowledge model $\mathcal{N} = \langle N_1, N_2, V_1 \cup V_2 \rangle$. Since I consist of positively determined clauses, it is straightforward to check that $\mathcal{N} \models I$. Since $\mathcal{N}_2$ is a minimal knowledge model for I, it follows that if $\psi \in \mathcal{L}_0$ and $\mathcal{N}_2 \models \psi$, then for every $\alpha \in N_1$, $\langle \mathcal{N}, \alpha \rangle \models \psi$. That is, $Th(\mathcal{N}_2) \cap \mathcal{L}_0 \subseteq Th(\mathcal{N}_1) \cap \mathcal{L}_0$.

By a symmetry argument, $Th(\mathcal{N}_1) \cap \mathcal{L}_0 \subseteq Th(\mathcal{N}_2) \cap \mathcal{L}_0$. Hence, $Th(\mathcal{N}_1) \cap \mathcal{L}_0 = Th(\mathcal{N}_2) \cap \mathcal{L}_0$. Since $\mathcal{N}_i$, $i = 1, 2$, are universal S5-models, $Th(\mathcal{N}_1) = Th(\mathcal{N}_2)$, that is, $T_1 = T_2$. □

Because of space restrictions we provide only rough sketches of the proofs of Theorems 4.4 and 5.2. In what follows we assume some familiarity with modal logics. The reader is referred to [HC84] and [Che80] for the comprehensive treatment of the subject.

Following Segerberg [Seg71], by an *index* we mean a finite sequence $\langle \epsilon_1, \ldots, \epsilon_n \rangle$, where $n > 0$, and for each i, $1 \le i \le n$, ϵ_i is a positive integer or ω (the first infinite ordinal). A frame $\langle M, R \rangle$ is said to be *determined by index* $\langle \epsilon_1, \ldots, \epsilon_n \rangle$ if:

1. M is a disjoint union of sets $M_1, \ldots, M_n$ such that for each i, $1 \le i \le n$, ϵ_i is the cardinality of M_i;
2. for each $\alpha, \beta \in M$, $\alpha R \beta$ if and only if $\alpha \in M_i$, $\beta \in M_j$ for some $1 \le i \le j \le n$.

Clearly, different indices determine different frames and all frames determined by the same index are isomorphic. A frame determined by an index is called an *index frame*. A logic $\mathcal{S}$ is *determined by an index a* if $\mathcal{S}$ is determined by the class of all index frames with index a. We will denote this logic by S_a. If $\{a_1, \ldots, a_n\}$ is a finite set of indices, by $S_{a_1,\ldots,a_n}$ we denote the logic $S_{a_1} \cap \ldots \cap S_{a_n}$.

Now, we have the following facts.

Proposition 8.1 *Logic* S5 *is characterized by the index frame with the index* $\langle \omega \rangle$. *Logic* S4F *is characterized by two index frames, one with index* $\langle \omega, \omega \rangle$ *and the other with index* $\langle \omega \rangle$.

Proposition 8.2 *Let* $\mathcal{S}$ *be a normal modal logic contained in* S5 *and properly containing* S4F. *Then, there are positive integers m and n such that* $S_{\langle \omega, m \rangle} \cap S_{\langle n, \omega \rangle} (= S_{\langle \omega, m \rangle, \langle n, \omega \rangle})$ *is contained in* $\mathcal{S}$.

Proof: Segerberg [Seg71] proved that each normal modal logic contained in S5 and containing S4F is the intersection of finitely many index logics. Thus, there are indices $a_1, \ldots, a_n$ such that $\mathcal{S} = S_{a_1} \cap \ldots \cap S_{a_n}$. Since axiom F may be falsified in any index frame determined by an index of length greater than 2, it follows that each index a_i, $1 \leq i \leq n$, has length at most 2. Now, the assertion follows from Proposition 8.1 and the three easy observations (the proofs can also be found in [Seg71]):
1. For each positive integer n, $S_{\langle\omega\rangle} \subseteq S_{\langle n\rangle}$;
2. For all m, n, where m and n stand either for a positive integer or ω, $S_{\langle m,n\rangle} \subseteq S_{\langle n\rangle}$;
3. Let each of m_1, m_2, n_1, n_2 denote a positive integer or ω. If $m_1 \leq m_2$ and $n_1 \leq n_2$, then $S_{\langle m_2,n_2\rangle} \subseteq S_{\langle m_1,n_1\rangle}$. □

The following result plays the crucial role in the proofs of Theorems 4.4 and 5.2.

Proposition 8.3 *Let $\mathcal{S} = S_{\langle n,\omega\rangle,\langle\omega,m\rangle}$. Let I be a theory consisting of the following $n+1$ formulas:*

$$\begin{array}{ll} \varphi_1 = & Kp_1 \supset p_2 \\ \varphi_2 = & K(p_1 \supset p_2) \supset p_3 \\ \ldots & \ldots \ldots \\ \varphi_{n+1} = & K(p_1 \wedge \ldots \wedge p_n \supset p_{n+1}) \supset p_1, \end{array}$$

where $p_1, \ldots, p_{n+1}$ are pairwise different propositional variables. Then the theories $ST(\emptyset)$ and $ST(p_1, \ldots, p_{n+1})$ are $\mathcal{S}$-expansions of I.

Proof: It is proved in [MST91] (see also [MT90]) that if a stable theory $ST(S)$ contains a theory I and $S \subseteq Cn_{\mathcal{S}}(I \cup \{\neg K\psi : \psi \notin ST(S)\})$, then $ST(S)$ is an $\mathcal{S}$-expansion for I. From this result, it follows immediately that $ST(\emptyset)$ is an $\mathcal{S}$-expansion for I.

The argument showing that $ST(p_1, \ldots, p_{n+1})$ is an $\mathcal{S}$-expansion for I is rather long but straightforward. The main idea is the following. Let $q_1, \ldots, q_{m+1}$ be a list of pairwise different propositional variables. Denote by ψ_1 the formula $\neg Kq_1$, and for $1 < i \leq m+1$, denote by ψ_i the formula $\neg K(q_1 \wedge \ldots \wedge q_{i-1} \supset q_i)$. It is straightforward to check that the formula

$$(K\alpha \wedge MKp_1 \wedge \ldots \wedge MK(p_1 \wedge \ldots \wedge p_n \supset p_{n+1}) \wedge K\psi_1 \wedge \ldots \wedge K\psi_{m+1}) \supset (p_1 \wedge \ldots \wedge p_{n+1}),$$

where α is the conjunction of all the formulas in I, is a theorem of $\mathcal{S}$ (it is sufficient to check that this formula is valid in each of the two index frames: $\langle n, \omega\rangle$ and $\langle\omega, m\rangle$).

It follows from our discussion that

$$\{p_1, \ldots, p_{n+1}\} \subseteq Cn_{\mathcal{S}}(I \cup \{\neg K\psi : \psi \notin ST(p_1, \ldots, p_{n+1})\}).$$

Since $I \subseteq ST(p_1, \ldots, p_{n+1})$, it follows that $ST(p_1, \ldots, p_{n+1})$ is an $\mathcal{S}$-expansion for I. □

Proof of Theorem 4.4: The assertion of the theorem follows directly from Proposition 8.2 and 8.3. □

Proof of Theorem 5.2: Let $\mathcal{S}$ be a modal logic properly containing S4F and contained in S5. Consider the default theory (D, W), where $W = \emptyset$ and D consists of $n+1$ defaults:

$$\frac{p_1 :}{p_2}, \ldots, \frac{p_1 \wedge \ldots \wedge p_n \supset p_{n+1} :}{p_1}.$$

Clearly, (D, W) has exactly one extension. On the other hand, $emb(D, W)$ is exactly the theory specified in Proposition 8.3. Hence, the theorem follows. □

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