

**Realizability Semantics  
for Error-Tolerant Logics  
(preliminary version)**

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*Abstract*

Classical and constructive logics have shortcomings as foundations for sophisticated automated reasoning from large amounts of data because a single error in the data could produce a contradiction, logically implying all possible conclusions. Relevance logics have the potential to support sensible reasoning from data that contains a few errors, limiting the impact of those errors to assertions that are naturally related to the erroneous information. There are a number of competing formal systems for relevance logic in the literature, with different sets of theorems. Applications of relevance logics, and particularly choices between formalisms, are hampered by the lack of clear intuitive semantic treatment of relevance.

This paper proposes plausible semantic treatments of relevance logic based on intuitive restrictions on the behavior of realizability functions. We examine two versions of realizability semantics. The first uses models which consist entirely of realizability functions that preserve *independence* of evidence, while the second semantics requires functions to be *strictly monotone* with respect to strength of evidence. We show soundness for the first semantics, and soundness and completeness theorems over a "nonstandard" set of models for the second. The second approach also yields completeness over "nonstandard" models for intuitionistic implication.

## 1. Introduction

Many computer systems, such as database query systems and “expert” systems, are designed to answer questions based on a given body of information:

Mr./Ms.  $X$  provides the computer with a body of information (or refers implicitly to information already stored) and asks a question.

The computer derives an answer and responds.

It is also natural to view programming in logic programming languages [Kow79], [ODo85] in this way. A formal framework for defining the correct behavior of these systems is well-known: the body of information is assumed to be in a particular logical language, the question gives rise to a set of *syntactically appropriate* answers, and the response is *correct* if it is both syntactically appropriate to the question and logically implied by the given information. (Often questions are of the form “*For what  $x$  is  $P(x)$ ?*,” in which case the syntactically appropriate answers are all formulas of the form  $P(y)$ ; see [BS76] for further discussion.) Assuming the notion of syntactic appropriateness to be rather straightforward, we see that correctness reduces to logical implication. Classical logic, in which assumptions  $p_1, \dots, p_n$  imply a conclusion  $c$  if every possible situation (world, model or interpretation) with all of the  $p_i$  true also has  $c$  true, is typically considered an appropriate basis for establishing correctness.

In relational database systems, the logical language expressing the information is a small subset of the first order predicate calculus. The only formulas are *tuples* such as  $Manages(Joe, Fred)$ , i.e., predicate symbols applied to sequences of constant symbols. Questions are of the form

*For what  $x_1, \dots, x_n$  is  $p[x_1, \dots, x_n]$ ?*

where  $p$  is a first-order formula over the constant and predicate symbols occurring in tuples. Since the form of the question is always the same, only the predicate  $p$  need be given; actual database systems provide some syntactic sugar for expressing predicates. The syntactically appropriate answers to such a relational question are the ground formulas of the form  $p[c_{1,1}, \dots, c_{1,n}]$ . A query processor is expected to find all correct answers. As with questions, the common structure of answers is omitted, and only the relevant tuples are generally given. Relational database query systems are designed to yield precisely the tuples representing formulae of the form described above which are logically implied by the information in the database. The semantics used to determine logical implication are classical first-order semantics, which seems to work quite well for simple query and answer. See [GMN84] for a survey of logical deduction as a database concept.

Relational database systems may be used as a paradigm for more sophisticated systems merely by allowing more and more generality in the formulae in the database. For example, the formula

$$\forall x, y, z. (Manages(x, z) \wedge Manages(y, z)) \rightarrow x = y$$

asserts the important fact that an employee may have only one manager. Such assertions are studied in the literature on relational databases under the name of *functional dependencies*, but they are treated as assertions *about* the contents of a database, and are not allowed to be *in* a database themselves.

Unfortunately, classical first-order predicate calculus has some shortcomings as a foundation for deriving the logical consequences of *unrestricted* first-order formula in a database. For example, suppose the formula asserting that no employee has more than one manager is stored in a database along with the two formulas  $Manages(Fred, Joe)$  and  $Manages(Alice, Joe)$  and the intuitively obvious formula  $Fred \neq Alice$ . Since these four formulae are mutually contradictory, there is no interpretation in which they are simultaneously true. Therefore, in classical first-order logic, *every* formula is a logical consequence of these four. So, according to a correctness criteria based on classical logic, a database systems answering

*For what  $x$  is  $Bonus(Mary, x)$ ?*

with \$1,000,000,000 is considered correct. Since this does not really seem intuitively correct, the classical correctness condition is too permissive. Without rejecting logic *per se* as a basis for correctness, we would like to be more restrictive. Since the erroneous inference above is based on irrelevant erroneous postulates (the uniqueness of managers is not relevant to Mary's bonus), a formal analysis of relevance seems to be required. Notice that the relational database model avoids the problems associated with logical contradiction by restricting the contents of databases in a way that makes it impossible to express logical contradiction (although information in a database may certainly be wrong, and may contradict other known information not included in the database).

The problem of reasoning in the presence of error is a problem of generating *knowledge* from other knowledge. Two considerations lead to a very different treatment of knowledge here than in the more usual approach through modal logic [Hin62]. The purpose of the modal approach is to provide a language for *discussing* knowledge. Thus, a special modal operator  $K$  is introduced, and  $Ka$  asserts that  $a$  is *known*. Assertions not preceded by  $K$  are taken as assertions of absolute truth. So  $Ka$  may be read as, *it is absolutely true that  $a$  is known*. In the database reasoning problem outlined above, *all* assertions are taken to be assertions of knowledge, based on the best available information, rather than of absolute truth, so an explicit modal operator is superfluous. Also, the usual modal definition of knowledge requires that knowledge is a refinement of truth. That is,  $Ka \rightarrow a$  (if  $a$  is known, then  $a$  is true) is a postulate. Such a definition of knowledge as realized truth is interesting theoretically, but has restricted application, since there is no general way in practice to distinguish true knowledge from rationally supported belief. So, we will define a logic in which the assertion  $a$  is taken to mean that  $a$  is known, or believed, on the basis of rational consideration of evidence from reliable (but not infallible) sources. We submit that this sort of assertion is the one normally intended by the phrase "it is known." The reader who is offended by such an interpretation is encouraged to substitute *rational belief* for *knowledge*, and to consider the following proposals accordingly.

Before discussing solutions to the problem of reasoning in the presence of contradictions and other errors, we would like to rule out two tempting but flawed approaches. First, we might try to detect and eliminate contradictions before answering any questions. Detection of contradiction is as hard as every other inference problem, and seems to require processing the whole database. There is no reason to hope for a tractable detection strategy except at the cost of very strong restrictions on a database. Once a contradiction is detected, it must be removed. Fagin, Ullman and Vardi discussed ways of removing contradictions by making syntactically minimal changes to a database [FUV83]. This work was intended as a foundation for *updates* to databases, but there is no evidence that syntactically minimal changes can be determined efficiently, nor that they are intuitively appropriate. If automated reasoning is to be applied to complex formulae, it appears unavoidable that we must survive contradictory and erroneous data until it is detected and corrected. In the meantime, we cannot avoid giving erroneous answers to questions that depend on the erroneous data, but we should answer other questions correctly.

Another natural approach to error is to quantify it through probability theory or some other numerical theory of uncertainty. It may be feasible to quantify some possibilities for error in such a way, but never all of them. No matter how thoroughly we have analyzed known uncertainties, we must accept the possibility of errors arising in totally unforeseen ways. Typographical errors alone (particularly errors in typing in probabilities of error) may be deadly. So, the quantification of error presupposes a foundation that can survive unquantified errors. We will not even introduce a nonnumerical modal operator to point out uncertainty (informally, a phrase of the form, "it is likely, but not certain, that ...") since *all* assertions are to be taken as potentially erroneous.

In order to reason safely in the presence of errors, we must change the basic attitude of classical logic toward logical implication. Classically, the postulate  $p$  implies the conclusion  $c$  if, whenever  $p$  is *absolutely, unquestionably true*, then so is  $c$ . In practical reasoning, we are never absolutely sure of postulates, and must be ready to abandon the classical viewpoint when necessary. In the past, this change of viewpoint has been accomplished by abandoning temporarily the formal system in which reasoning has been going on, and switching to a more conservative informal system. For example, when set theory was discovered to be inconsistent, mathematicians were able to distinguish informally those inferences

that depended on the contradiction from those that did not (even though most of these mathematicians professed classical ideology).<sup>1</sup> For automated reasoning, the more conservative viewpoint that survives contradictions must itself be formalized. Notice that intuitionistic, or constructive, reasoning does not by itself solve the problem. To an intuitionist,  $p$  implies  $c$  if, whenever  $p$  may be *absolutely, unquestionably demonstrated* by some effective procedure, then so may  $c$ . Intuitionistic logic rejects certain classically accepted formulae, such as  $a \vee \neg a$ , but still allows a contradiction to imply all possible conclusions. The new sort of implication required lets  $p$  imply  $c$  if, whenever the best available information includes  $p$ , it is reasonable to believe  $c$  as well. It is not immediately obvious how to formalize such a concept, but the example of uniqueness of managers implying Mary's large bonus shows that neither classical nor intuitionistic formal logic suffices. Apparently, the new implication must enforce some sort of *relevance* of  $p$  to  $c$  when  $p$  implies  $c$ .

Logicians of a philosophical bent have studied several formal systems enforcing relevance of postulates to implied conclusions. Such systems are collectively called *relevance logics*, and are best explained by Anderson and Belnap [AB75]. Proponents of relevance logic from the philosophical community often claim that some sort of relevance logic represents the only true idea of logical implication. For our purposes, it is only important that a relevance logic may provide a useful definition of logical implication in the presence of error. Shapiro and Wand [SW76], and Belnap [Bel76] [Bel75], have already advocated the usefulness of relevance logics to automatic reasoning.

The application of relevance logics has been hindered by a lack of intuitively satisfying semantics for such logics. Routley and Meyer [RM73] and, independently, Fine [Fin74], defined Kripke-like semantics for certain relevance logics, and proved soundness and completeness. Their semantics involves a *ternary* relation between possible worlds, and that relation seems very difficult to understand intuitively.

This paper proposes new semantic treatments of relevant implication, based on the intuitionistic concept of realizability [Kle45, KV65, Lau70]. Soundness and completeness theorems are proved for relevant implication, motivated by similar theorems for intuitionistic implication. The new semantics may provide a plausible foundation for the formal analysis of error-tolerant reasoning. Avron recently and independently proposed a totally different semantic explanation of another relevance logic. His approach divides the universe into domains of relevance; classical logic is valid within, but not between, domains. For Avron,  $p$  relevantly implies  $c$  if  $\neg p \vee c$ , and  $p$  is in the same domain of relevance as  $c$ . By contrast, our semantics define the relation  $p$  relevantly implies  $c$ , but not any symmetric relation of relevance between predicates. There is not enough technical information about applications of the two semantic treatments yet to allow a rational choice between them.

Section 2 sketches the concepts behind realizability semantics for relevance logics. Section 3 introduces a well-known formal system for intuitionistic implication, based on the lambda- $K$  calculus, and two formal systems for relevant implication, derived from the intuitionistic system by restricting the use of variables in terms (equivalently, the use of assumptions in proofs). Section 4 presents realizability semantics for intuitionistic implication, with soundness and completeness. Section 5 extends the semantics, soundness results, and a completeness result, to relevance logics.

## 2. Intuitive Foundations for Relevance Logic

Although intuitionistic logic does not allow meaningful reasoning in the presence of contradictions, intuitionistic semantics seems to provide a useful starting point for developing relevant semantics. A constructive proof of  $a \rightarrow b$  is an effective function mapping constructive proofs of  $a$  to constructive proofs of  $b$ . In the presence of errors, such a function must map not only proofs of  $a$ , but seemingly reliable *evidence* for  $a$  to at least as reliable evidence for  $b$ . See Helman [Hel77b] for a discussion of the philosophical aspects of proofs as functions on evidence. This view of proofs as functions leads naturally to a consideration of types: if we regard the collection of evidence for  $a$  as the "type"  $a$ , and

<sup>1</sup>However, at least one false theorem was published by Burali-Forti [Bur67], who did *not* recognize his derivation as a paradox, but thought that it proved that the ordinals are not well ordered. (Set theorists today uniformly agree that the ordinals are well ordered.)

similarly for  $b$ , then a proof of  $a \rightarrow b$  is a function from type  $a$  to type  $b$ . Serendipitously, the type of functions from  $a$  to  $b$  is commonly written  $a \rightarrow b$ .

The interpretation of proofs as functions certainly supports *modus ponens*, from  $a$  and  $a \rightarrow b$  to infer  $b$ , since the function proving  $a \rightarrow b$  can be applied to the evidence for  $a$ , producing evidence for  $b$ . Problems may arise when evidence *against*  $b$  is taken as evidence against  $a$ . Suppose that the function proving  $a \rightarrow b$  ignores its input, and produces a fixed piece of seemingly reliable, but erroneous, evidence for  $b$ . Suppose that evidence against  $b$  is also available. It makes no sense to take this evidence against  $b$  as evidence against  $a$ , since the proof that  $a \rightarrow b$  did not actually establish any connection between  $a$  and  $b$ .

The standard intuitionistic development of negation (also valid classically) provides a concrete illustration of the problem of negative evidence.  $\neg a$  is defined to be  $a \rightarrow \Lambda$ , where  $\Lambda$  is a manifest falsehood, and  $\Lambda \rightarrow b$  for all  $b$ . As discussed in Section 1, a contradiction  $a \wedge \neg a$  with such a negation implies all formulae  $b$ . Furthermore, the constructive interpretation of implication, in the presence of errors, does not support *modus tollens*. *Modus tollens*, or contrapositive reasoning, allows  $\neg a$  to be inferred from  $a \rightarrow b$  and  $\neg b$ . *Modus tollens* may be derived from *modus ponens*, and the definitions of  $\neg$  and  $\Lambda$  above. But, as argued before, if the constructive proof of  $a \rightarrow b$  is a constant function, evidence against  $b$  does not yield evidence against  $a$ . Intuitionistic logic is normally interpreted over a universe containing only correct constructive proofs. In that universe, *modus tollens* presents no problem. We seek a modified interpretation of implication over a universe containing both correct and incorrect evidence, that still supports *modus tollens*.

In addition to expanding the domains of functions to include evidence as well as proof, we must require a function proving  $a \rightarrow b$  to establish a real connection between  $a$  and  $b$ . The stronger concept of implication, with an appropriate definition of negation, should support *modus tollens* as well as *modus ponens* in spite of errors. In particular, a relevant proof that  $a \rightarrow b$  must be a function that, given more and more evidence for  $a$ , produces more and more evidence for  $b$ . With such an increasing property, any reason for distrusting evidence for  $b$  gives a legitimate reason to distrust evidence for  $a$  as well. Constant functions must certainly be ruled out, but many nonconstant functions, for instance those that are constant over large subsets of their domains, must be ruled out as well. We formalize these intuitive concerns in two ways by imposing a relation of *independence* between pieces of evidence, or by imposing an anti-symmetric ordering relation, and insisting in each case that proofs of relevant implication preserve the relation.

### 3. Proof Systems for Intuitionistic and Relevance Logics

Type-theoretic foundations for constructive logic originated in [Kre56], and may also be found in [How80, Ste72, FLO83, Mar75]. We will take a type-theoretic approach to both proofs and semantics of formulas, using the notation of [Ste72, FLO83]. The basic syntactic idea is to adopt the *typed lambda calculus*, a notation for describing typed functions, as a proof system. The usual typed lambda calculus (lambda-K calculus) serves as a proof system for intuitionistic logic. Proof systems for various relevance logics may be derived from lambda-K calculus by imposing restrictions which insure that all arguments to a function are relevant to the value of the function. The most obvious relevance restriction is that the argument of a function must appear in the body of the function definition. This restriction yields the *lambda-I calculus* and the relevance system **R**. Additional restrictions, motivated by semantic definitions for relevant functions, yield other relevance logics.

It is helpful to think of lambda terms as natural deduction proofs since lambda calculus is actually a notational variant of natural deduction [How80, Mar75, MP85]. Natural deduction proofs are intended to formalize the common “blackboard-style” arguments in which we assume  $\alpha$ , derive  $\beta$  from  $\alpha$ , and then conclude from this derivation that  $\alpha$  must imply  $\beta$ . Thus natural deduction proof involve introducing assumptions into proofs and later discharging assumptions to obtain proofs of implications. In lambda calculus expressions, which we will call *testimony terms*, assumptions are represented by variables and discharged assumptions by lambda-bound variables. Since we will consider a formula  $a$  proved only when

we have a proof of  $a$  with all assumptions discharged, our proof system will involve some machinery for keeping track of free variables in terms.

In the following definitions,  $\vdash$  and  $\rightarrow$  are intended as formal symbols, while  $\in$ ,  $\implies$ , and  $\longrightarrow$  are metasympols denoting the usual mathematical concepts of set membership, implication, and sets of functions.

**Definition 3.1**

Let  $\mathbf{P}_0$  be a set of symbols called *primitive types or propositions*. Members of  $\mathbf{P}_0$  (and later  $\mathbf{P}$ ) are denoted  $a, b, c, \dots$

The set  $\mathbf{P}$  of *types or propositions* is defined inductively as follows:

- P1.  $a \in \mathbf{P}_0 \implies a \in \mathbf{P}$
- P2.  $a, b \in \mathbf{P} \implies (a \rightarrow b) \in \mathbf{P}$  ■

**Definition 3.2**

For each  $a \in \mathbf{P}$ , we choose an infinite set  $\mathbf{V}^a$  of *variables of type  $a$* , written  $u^a, v^a, \dots$ , and let  $\mathbf{V}$  be the set of all variables  $\mathbf{V} = \bigcup \{ \mathbf{V}^a \mid a \in \mathbf{P} \}$ .

The set  $\mathbf{T}$  of *testimony terms*, together with their types, are defined inductively below. Members of  $\mathbf{T}$  are denoted  $\alpha, \beta, \gamma, \dots$ . We write  $\alpha \vdash a$  to mean that the testimony term  $\alpha$  has type  $a$  or, equivalently,  $\alpha$  is a proof of  $a$ .

- T1.  $x^a \in \mathbf{V}^a \implies x^a \in \mathbf{T} \wedge x^a \vdash a$
- T2.  $\alpha, \beta \in \mathbf{T} \wedge \alpha \vdash a \wedge \beta \vdash (a \rightarrow b) \implies (\beta\alpha) \in \mathbf{T} \wedge (\beta\alpha) \vdash b$
- T3.  $\beta \in \mathbf{T} \wedge \beta \vdash b \wedge x^a \in \mathbf{V}^a \implies (\lambda x^a \beta) \in \mathbf{T} \wedge (\lambda x^a \beta) \vdash (a \rightarrow b)$

The set  $\mathcal{A}(\alpha)$  of assumptions, or free variables, in term  $\alpha$  is defined as follows.

- F1.  $\mathcal{A}(x^a) = \{x^a\}$
- F2.  $\mathcal{A}(\beta\alpha) = \mathcal{A}(\beta) \cup \mathcal{A}(\alpha)$
- F3.  $\mathcal{A}(\lambda y^b \beta) = \mathcal{A}(\beta) - \{y^b\}$  ■

The testimony terms above are the terms of the conventional typed lambda-K calculus. It is easy to see that if  $\alpha \in \mathbf{T}$ , then there is a unique  $a$  such that  $\alpha \vdash a$ . If  $\mathcal{A}(\alpha) = \emptyset$ , we say  $\alpha$  is *closed*. An occurrence of variable  $x^a$  within a subterm  $\lambda x^a \alpha$  is said to be *bound*.

For one plausible definition of relevance, a more subtle analysis of the occurrence of variables is required.

**Definition 3.3**

Let  $X \subseteq \mathbf{V}$  be a set of variables. The set  $\mathcal{R}_X(\alpha)$  of variables occurring *relevantly in  $\alpha$  with respect to  $X$*  is defined as follows.

- RR1.  $\mathcal{R}_X(x^a) = \{x^a\}$
- RR2.  $\mathcal{R}_X(\beta\alpha) = \mathcal{R}_X(\beta) \cup (\mathcal{R}_X(\alpha) - \mathcal{A}(\beta))$
- RR3.  $\mathcal{R}_X(\lambda y^b \beta) = \mathcal{R}_{X \cup \{y^b\}}(\beta) - \{y^b\}$  ■

The condition  $\alpha$  is *relevant in  $x^a$  with respect to  $X$*  is equivalent to  $\alpha$  is *free in  $x^a$ , and no bound variable or variable in  $X$  appears to the left of all occurrences of  $x^a$* . Notice that if  $\alpha$  is relevant in  $x^a$  with respect to  $X$  and  $Y \subseteq X$ , then  $\alpha$  is relevant in  $x^a$  with respect to  $Y$ . In particular,  $\alpha$  is relevant in  $x^a$  with respect to  $\emptyset$  if and only if  $\alpha$  is *free in  $x^a$ , and no bound variable of  $\alpha$  appears to the left of all occurrences of  $x^a$* . Two interesting classes of relevant terms may be defined by restricting abstractions  $(\lambda x^a \beta)$  so that  $x^a$  must be free (alternatively, relevant) in  $\beta$ .

**Definition 3.4**

The restricted testimony set  $\mathbb{T}_R$  is defined as  $\mathbb{T}$ , except that the last clause is changed to

$$\mathbb{T}3'. \beta \in \mathbb{T} \wedge \beta \vdash b \wedge x^a \in \mathcal{A}(\beta) \implies (\lambda x^a \beta) \in \mathbb{T} \wedge (\lambda x^a \beta) \vdash (a \rightarrow b)$$

Similarly,  $\mathbb{T}_{RR}$  is defined as  $\mathbb{T}$ , except that the last clause is changed to

$$\mathbb{T}3''. \beta \in \mathbb{T} \wedge \beta \vdash b \wedge x^a \in \mathcal{R}_\theta(\beta) \implies (\lambda x^a \beta) \in \mathbb{T} \wedge (\lambda x^a \beta) \vdash (a \rightarrow b)$$

We use  $\alpha \vdash_R b$  as an abbreviation for  $\alpha \vdash b \wedge \alpha \in \mathbb{T}_R$ , and similarly for  $\vdash_{RR}$ . ■

The lambda terms in the testimony set  $\mathbb{T}_R$  defined above are precisely the terms of the *simply typed lambda-1 calculus* [FLO83], [Ste72], [Chu41]. The set  $\mathbb{T}_{RR}$  has the additional restriction that the left-to-right order of variables abstracted by nested  $\lambda$ 's must be the same as the left-to-right order of their leftmost occurrences. For example,  $xy(xzy)$  may only be abstracted as  $\lambda x \lambda y \lambda z (xy(xzy))$ , not as  $\lambda y \lambda x \lambda z (xy(xzy))$ , etc.

Lambda terms are usually intended to be read as function definitions, with  $(\beta\alpha)$  representing the function  $\beta$  applied to the argument  $\alpha$ , and  $\lambda x^a \alpha$  representing the function of  $x^a$  whose values are described by the term  $\alpha$ . We follow the usual conventions for the lambda calculus [Chu41]: parentheses are omitted with the convention that application associates to the left and  $\rightarrow$  associates to the right. Superscripts are omitted from variables when the type is clear from context, or not important. Since the functional nature of lambda terms is reflected in the computation rules of  $\beta$ - and  $\eta$ -reduction, restricted class of lambda terms are unlikely to semantically define restricted classes of functions unless they are closed under these reduction rules. It is easy to show that  $R$  and  $RR$  are closed under  $\beta$ - and  $\eta$ -reduction.

**Theorem 3.5**

Let  $Q$  be  $R$ ,  $RR$ , or null.

If  $\alpha[(\lambda x^a \beta[x^a])\gamma] \vdash_Q b$ , then  $\alpha[\beta[\gamma]] \vdash_Q b$ .

If  $\alpha[(\lambda x^a (\beta x^a))] \vdash_Q b$ , and  $x^a \notin \mathcal{A}(\beta)$ , then  $\alpha[\beta] \vdash_Q b$ .

Proof sketch: elementary induction on the structure of  $\beta$ . ■

Notice that  $\mathbb{T}_R$  and  $\mathbb{T}_{RR}$  are *not* closed under the reverse of  $\beta$ - and  $\eta$ -reduction.

Having defined the syntax of testimony terms, we may gain some intuition for lambda calculus as a proof system by informally translating testimony terms into natural deduction proofs (see [How80, FLO83, MP85, Ste72] for further discussion). If  $\beta \vdash a \rightarrow b$  and  $\alpha \vdash a$ , then  $(\beta\alpha)$  denotes the proof

*the proof denoted by  $\beta$ , ending in*

$$i_1 : a \rightarrow b$$

*the proof denoted by  $\alpha$ , ending in*

$$i_2 : a$$

$b$  by Modus Ponens,  $i_1, i_2$

Similarly, if  $\beta \vdash b$ , then  $\lambda x^a \beta$  denotes the proof

Assume  $a$

*the proof denoted by  $\beta$ , ending in*

$$i : b$$

end assumption

$a \rightarrow b$  by discharging assumption  $a$

For example, the lambda term for the identity function,  $(\lambda x^a x^a)$ , may be read as a proof that  $a$  implies  $a$ :

1 : Assume  $a$

2 :  $a$  by 1

end assumption

3 :  $a \rightarrow a$  by Deduction Rule, 2

Since terms with free variables correspond to partial proofs with undischarged assumptions, the theorems of the systems  $\mathbf{I}_\rightarrow$ ,  $\mathbf{R}_\rightarrow$ , and  $\mathbf{RR}_\rightarrow$  are those types  $a$  for which there exists a closed term with  $\alpha \vdash a$ ,  $\alpha \vdash_R a$ , and  $\alpha \vdash_{RR} a$ , respectively:

**Definition 3.6**

$\vdash_Q a \iff \exists \alpha \in \mathbf{T}_Q . \alpha \vdash_Q a \wedge \mathcal{A}(\alpha) = \emptyset$ , where  $Q$  is  $R$ ,  $RR$ , or null

$\mathbf{I}_\rightarrow = \{a \mid \vdash a\}$ ,  $\mathbf{R}_\rightarrow = \{a \mid \vdash_R a\}$ ,  $\mathbf{RR}_\rightarrow = \{a \mid \vdash_{RR} a\}$ , ■

$\mathbf{I}_\rightarrow$  is the positive intuitionistic theory of implication.  $\mathbf{R}_\rightarrow$  is the theory of relevant implication from [AB75], syntactically the most natural logic of relevant implication. Clearly,  $\mathbf{RR}_\rightarrow \subseteq \mathbf{R}_\rightarrow \subseteq \mathbf{I}_\rightarrow$ . Both containments are strict, the first since  $(\lambda x^a \lambda y^{a \rightarrow b} (yx)) \vdash_R a \rightarrow (a \rightarrow b) \rightarrow b$ , but not  $\vdash_{RR} a \rightarrow (a \rightarrow b) \rightarrow b$ , (although  $(\lambda y^{a \rightarrow b} \lambda x^a (yx)) \vdash_{RR} (a \rightarrow b) \rightarrow a \rightarrow b$ ). The second containment is strict because  $(\lambda x^a \lambda y^b x) \vdash a \rightarrow b \rightarrow a$ , but not  $\vdash_R a \rightarrow b \rightarrow a$ . The intuitionistic theory of implication,  $\mathbf{I}_\rightarrow$ , is itself a strict subtheory of the classical theory of implication since  $((a \rightarrow b) \rightarrow a) \rightarrow a$  holds classically, but not  $\vdash ((a \rightarrow b) \rightarrow a) \rightarrow a$ . The restricted relevance system  $\mathbf{RR}_\rightarrow$ , motivated by semantic considerations discussed in Section 5.1, seems to be new. The restriction on binding order is similar to, but more stringent than, that of *ticket entailment* ( $\mathbf{T}_\rightarrow$  of [AB75]). Proofs of ticket entailments are usually defined by restricting application (*modus ponens*), rather than lambda-abstraction (*the deduction rule*). Each of  $\mathbf{RR}_\rightarrow$  and  $\mathbf{T}_\rightarrow$  may be presented either way. The difference does not affect the theorems, but does change the proofs from postulates.

The basic idea of connecting relevance logics to the lambda-I calculus has been noticed before. In particular, Helman [Hel77a] gave an insightful treatment of a lambda-I calculus, extended to handle conjunction, and showed a restricted soundness and completeness of the typing mechanism for a particular semantic interpretation. Helman's definitions of soundness and completeness were *not* intended to provide thorough foundations for reasoning. Helman's semantics depend on the inclusion in each functional domain of a special element representing undefinedness. Helman's relevant functions are the strict functions – those that preserve undefinedness – including functions that are constant on the well-defined elements of their domains. Since the undefined elements are not denotable by lambda terms, undefined values may only be produced by returning an undefined argument, so strict constant functions, and many other conceivable functions with undefined elements, are not lambda-definable. Helman did not try to analyze the behavior of the definable strict functions on well-defined arguments, which we believe is crucial to the intuitive impact of such semantics. Rather than perform such an analysis, we have sought a direct characterization of the extensional character of relevant functions, without adding any intensional structure such as undefined elements. Helman's completeness depends critically on the impossibility of producing an undefined value except by copying out an undefined argument, and also on the *a priori* assumption that all functions of interest as potential proofs are lambda-definable. Pottinger [Pot74] also defined relevant implication in an intuitionistic setting, and investigated the interaction of the two implications. He did not attempt to provide semantics.

## 4. Evidence Semantics for Intuitionistic Logic

### 4.1. Evidence

Let a particular language be fixed for the rest of the paper, *i.e.*, fix  $\mathbf{P}_0$ . Also fix an infinite set  $\mathcal{U}_0$  to be the universe of primitive objects. All of the subsequent discussion holds independently of these choices. A model for any of the logics of this paper consists of a set of evidence for each formula. The definition of model uses the preliminary definition of frame [Hen50].

**Definition 4.1**

A *frame*  $\mathcal{F}$  over  $\mathcal{U}_0$  is a family of sets  $\{\mathcal{F}_a \mid a \in \mathbf{P}\}$  indexed by types such that for all  $a \in \mathbf{P}_0$ ,  $\mathcal{F}_a \subseteq \mathcal{U}_0$ , and for all  $a, b \in \mathbf{P}$ ,  $\mathcal{F}_{a \rightarrow b}$  is a set of functions from  $\mathcal{F}_a$  to  $\mathcal{F}_b$ .

The *full frame*  $\{\mathcal{U}_a \mid a \in \mathbf{P}\}$  over  $\mathcal{U}_0$  is defined by

- U1.  $a \in \mathbf{P}_0 \implies \mathcal{U}_a = \mathcal{U}_0$
- U2.  $a, b \in \mathbf{T} \implies \mathcal{U}_{a \rightarrow b} = \{f \mid f : \mathcal{U}_a \rightarrow \mathcal{U}_b\}$

We use  $f, g, h, \alpha, \beta, \dots$  to denote members of a frame  $\mathcal{F}$ . ■

While an arbitrary frame may have an arbitrary selection of functions in  $\mathcal{F}_{a \rightarrow b}$ , a model must at least contain all functions defined by testimony terms.

**Definition 4.2**

A *lambda-K model* is a frame  $\mathcal{F}$  such that, for all  $a, b, c \in \mathbf{P}$ , there are elements  $K_{a,b} \in \mathcal{F}_{a \rightarrow b \rightarrow a}$  and  $S_{a,b,c} \in \mathcal{F}_{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c}$  with

$$K_{a,b}(\alpha)(\beta) = \alpha$$

$$S_{a,b,c}(\alpha)(\beta)(\gamma) = \alpha(\gamma)(\beta(\gamma))$$

A *fully inhabited model* is a frame  $\mathcal{F}$  with  $\mathcal{F}_a \neq \emptyset$  for all  $a \in \mathbf{P}$ .

The full frame  $\mathcal{U}$  over  $\mathcal{U}_0$  is also called the *standard model* over  $\mathcal{U}_0$ . ■

The lambda-K models are precisely the frames that are closed under lambda definability [Bar84, Mey82]. Note that the standard model is a lambda-K model. It seems worthwhile to mention an alternative to the above definitions. At first glance, a frame  $\mathcal{F}$  over  $\mathcal{U}_0$  might appear to be substructure of the full frame  $\mathcal{U}$ . This is not true because, although  $\mathcal{F}_{a \rightarrow b}$  is a subset of  $\mathcal{U}_{a \rightarrow b}$  for  $a, b \in \mathbf{P}_0$ ,  $\mathcal{F}_{a \rightarrow b \rightarrow b}$  is not a subset of  $\mathcal{U}_{a \rightarrow b \rightarrow b}$  since the functionals in  $\mathcal{U}_{a \rightarrow b \rightarrow b}$  are defined on a larger domain. However, as shown in the equational completeness proof of [Fri75], there is a partial homomorphism from the full frame  $\mathcal{U}$  over  $\mathcal{U}_0$  to any frame  $\mathcal{F}$  over  $\mathcal{U}_0$ . Since, for the purposes of obtaining completeness theorems for logics we are not specifically concerned with equations between testimony terms, we could have defined models as “substructures” of the full frame which contain  $K$  and  $S$  instead of as homomorphic images of substructures. This would make some technical arguments simpler, but complicates the interpretation of testimony terms as elements of models. Specifically, since extensionality fails on “substructures,” it is not clear which function to assign to a testimony term  $\lambda x^a. \alpha$  (see [Mey82] for further discussion). Furthermore,  $\eta$ -reduction fails on “substructures.” Läuchli proves completeness using substructures of the standard model [Lau70] and, in the final paper, we intend to use partial homomorphisms to compare his results with ours.

We associate semantic evidence with proofs according to the usual semantics of the typed lambda calculus.

**Definition 4.3**

An *environment*  $\eta$  for frame  $\mathcal{F}$  is a function from  $\mathbf{V}$  to  $\bigcup \mathcal{F}$ , such that  $\forall x^a \in \mathbf{V}^a. \eta(x^a) \in \mathcal{F}_a$ .

$\mathbf{Env}(\mathcal{F})$  denotes the set of all environments for frame  $\mathcal{F}$ .

If  $\eta \in \mathbf{Env}(\mathcal{F})$ , then  $\eta[\alpha/x^a]$  is the environment with  $\eta[\alpha/x^a](x^a) = \alpha$  and  $\eta[\alpha/x^a](y^b) = \eta(y^b)$  when  $x^a \neq y^b$

The *meaning* of a term  $\alpha \in \mathbf{T}$  in environment  $\eta$ , written  $\llbracket \alpha \rrbracket \eta$ , is defined by

$$\llbracket x^a \rrbracket \eta = \eta(x^a)$$

$$\llbracket (\beta \alpha) \rrbracket \eta = \llbracket \beta \rrbracket \eta(\llbracket \alpha \rrbracket \eta)$$

$$\llbracket ((\lambda x^a \beta)) \rrbracket \eta(f) = \llbracket \beta \rrbracket \eta[f/x^a] \text{ for all } f \in \mathcal{F}_a.$$

■

It can be shown that for every testimony term (lambda-term)  $\alpha$  and environment  $\eta$  in lambda- $K$  model  $\mathcal{F}$ , the meaning  $\llbracket \alpha \rrbracket \eta$  is well-defined [Bar84, Mey82]. In addition, if  $\alpha \vdash a$ , then  $\llbracket \alpha \rrbracket \eta \in \mathcal{F}_a$ . If  $\alpha$  is closed, then  $\llbracket \alpha \rrbracket \eta$  is independent of  $\eta$ , so we often write  $\llbracket \alpha \rrbracket$  for the meaning of  $\alpha$ .

One important model is the *term model* for an equational theory  $\mathcal{E}$ .

#### Definition 4.4

Let  $\mathcal{E}$  be an equational theory over  $\mathbf{T}$ , with  $=_{\mathcal{E}}$  the provable equality relation on terms.

The *term model* is the fully inhabited lambda- $K$  model  $\mathcal{T}^{\mathcal{E}} = \{\mathcal{T}_a^{\mathcal{E}}\}$ , where  $\mathcal{T}_a^{\mathcal{E}} = \{[\alpha] \mid \alpha \vdash a\}$ , with

$$\alpha \vdash a \in \mathbf{P}_0 \implies [\alpha] = \{\beta \mid \alpha =_{\mathcal{E}} \beta\}$$

$$\beta \vdash a \rightarrow b \implies [\beta] \text{ is the function defined by } \forall \alpha \vdash a. [\beta]([\alpha]) = [\beta\alpha]$$

$\mathcal{T}^{\beta\eta}$  denotes the term model for  $\beta, \eta$ -equality [Bar84, Fri75, Mey82, Sta82a]. ■

In the term model  $\mathcal{T}^{\mathcal{E}}$ , we have  $\alpha =_{\mathcal{E}} \beta$  iff  $[\alpha] = [\beta]$ . It is important to note that a term model  $\mathcal{T}^{\mathcal{E}}$  has elements  $[\alpha]$  defined by *open* terms as well as closed terms. In fact, since  $[x^a] \in \mathbf{T}_a$ ,  $\mathcal{T}^{\mathcal{E}}$  contains evidence for every formula. However, as we shall see, only certain formulas will have evidence in all beliefs. There is a lambda- $K$  model derived from the *closed* terms, in which each type with a closed term contains exactly one function, and each type without a closed term is empty. However, this model is not fully inhabited.

#### 4.2. Intuitionistic Belief and Validity

We define semantic validity using the subsidiary notion of belief. Beliefs are sets of evidence (from some model) which satisfy some closure conditions and the valid formulas in a model are taken to be those that are supported by all beliefs. Thus the validity of formulas is determined by the closure conditions used to define beliefs. The simplest notion of belief is a set of evidence that is closed under *modus ponens*.

#### Definition 4.5

A *belief*  $\mathcal{G}$  over frame  $\mathcal{F}$  is a family of sets  $\{\mathcal{G}_a\}$  such that

$$\mathcal{G}_a \subseteq \mathcal{F}_a$$

$$\forall a, b \in \mathbf{T}, \beta \in \mathcal{G}_{a \rightarrow b}, \alpha \in \mathcal{G}_a. \beta(\alpha) \in \mathcal{G}_b$$
 ■

This seems a minimal requirement on beliefs: if anyone believes evidence  $\beta \in \mathcal{F}_{a \rightarrow b}$  for the implication  $a \rightarrow b$ , and evidence  $\alpha \in \mathcal{F}_a$  for  $a$ , we assume that he or she will believe evidence  $\beta(\alpha)$  for  $b$ .

A reasonable belief for a particular logic should contain only functions that provide acceptable evidence for the logic in question, and a standard belief should include all such evidence. Different choices of sets of beliefs, as well as different choices of classes of models, will support different logical theories. For intuitionistic logic, we would like for beliefs to contain all of the *uniformly constructible functions*.

#### Pseudodefinition 4.6

A *standard intuitionistic belief* is a subframe of a frame  $\mathcal{F}$  containing exactly the uniformly constructible functions in  $\mathcal{F}$ .

A *full intuitionistic belief* is a subframe of a frame  $\mathcal{F}$  containing all of the uniformly constructible functions in  $\mathcal{F}$ , and perhaps more. ■

The pseudodefinition above *cannot* be formalized satisfactorily, since there is no proper formal characterization of the uniformly constructible functions. Fortunately, we can formally characterize classes of beliefs in such a way that the fullness of the beliefs is intuitively apparent, and the implicational theory of the formal classes of beliefs is the same as that of the standard beliefs. In particular, if a function is *uniformly* constructible, it cannot refer to the internal structure of the primitive objects that it operates on. Therefore, the function must be invariant under substitutions of one primitive object for another of the same type.

**Definition 4.7**

A *functionally closed belief* is a belief  $\mathcal{G}$  over frame  $\mathcal{F}$  such that

$$\mathcal{G}_{a \rightarrow b} = \{ \beta \in \mathcal{F}_{a \rightarrow b} \mid \forall \alpha \in \mathcal{G}_a . \beta(\alpha) \in \mathcal{G}_b \}$$

The *classical belief* over frame  $\mathcal{F}$  is  $\mathcal{F}$  itself.

A *hereditary permutation*  $\pi$  over frame  $\mathcal{F}$  is a family  $\{\pi_a\}$  of permutations defined as follows.

- $\pi 1.$   $a \in \mathbb{P}_0 \implies \pi_a(\alpha)$  is some permutation of  $\mathcal{F}_a$
- $\pi 2.$   $\pi_{a \rightarrow b} : \mathcal{F}_{a \rightarrow b} \rightarrow \mathcal{F}_{a \rightarrow b}$  is the permutation defined by  $\pi_{a \rightarrow b}(\alpha) = \pi_b \circ \alpha \circ \pi_a^{-1}$

The *invariant belief* over a frame  $\mathcal{F}$  is the belief containing  $\alpha \in \mathcal{F}_a$  iff  $\pi_a(\alpha) = \alpha$  for every hereditary permutation  $\pi$ . ■

Classical beliefs are always functionally closed, but invariant beliefs may not be. Invariant beliefs seem to be full intuitionistic beliefs over some models (see discussion of “P-structures” in [Sta82a]), but this is not susceptible to formal statement and proof. In lambda calculus parlance, a functionally-closed belief is called a *logical predicate*; see [Plo80, Sta82c].

The semantics for intuitionistic and relevance logics hinge on distinguishing certain evidence as *logical evidence*. We define a formula (type) to be *valid* in a model if the model contains logical evidence for the formula (logical evidence of that type). *Logical evidence* in a model is defined to be the evidence occurring in every legitimate belief. Several different definitions of legitimate belief can be considered, based on Definition 4.7 above. The intent is that a formula should be valid if there exists ideal abstract evidence for it in every model. Every proof is a construction of evidence, and we can ask whether the formally constructible evidence corresponds appropriately to the ideal evidence.

**Definition 4.8**

If  $\mathcal{F}$  is a model, and  $\mathcal{G}$  a belief,  $\mathcal{F}, \alpha, \mathcal{G} \models a$  is intended to mean that  $\alpha$  is evidence for  $a$  in  $\mathcal{F}$  and believed by  $\mathcal{G}$ . Formally,

$\mathcal{F}, \alpha, \mathcal{G} \models a$  if  $\alpha \in \mathcal{G}_a$

$\mathcal{F}, \alpha \models_F a$  if  $\mathcal{F}, \alpha, \mathcal{G} \models a$  for every functionally closed belief  $\mathcal{G}$

$\mathcal{F}, \alpha \models_C a$  if  $\mathcal{F}, \alpha, \mathcal{G} \models a$  for the classical belief  $\mathcal{G}$

$\mathcal{F}, \alpha \models_I a$  if  $\mathcal{F}, \alpha, \mathcal{G} \models a$  for the invariant belief  $\mathcal{G}$

$\mathcal{F} \models_Q a$  if there exists an  $\alpha$  with  $\mathcal{F}, \alpha \models_Q a$ , where  $Q$  is either  $F$ ,  $C$ , or  $I$

$\models_{S,Q} a$  if  $\mathcal{F} \models_Q a$  for every standard model  $\mathcal{F}$ , where  $Q$  is either  $F$ ,  $C$ , or  $I$

Similarly,  $\models_{\lambda K, Q}$  denotes validity in all lambda-K models,  $\models_{F\lambda K, Q}$  denotes validity in all fully inhabited lambda-K models. ■

Notice that the existential quantification over evidence comes before the universal quantification over beliefs. A set of beliefs in a model is used to distinguish the “logical evidence,” the evidence occurring in all belief, from arbitrary evidence. In general, the valid formulae are determined by the intersection of a set of beliefs. It is interesting to note that  $\models_{S,C}$  is classical validity [Lau65, Lau70]. Intuitionistic logic  $\mathbb{I}_-$  is sound for every combination, but we prove completeness only for  $\models_{\lambda K, F}$ ,  $\models_{F\lambda K, F}$ . Läuchli has proved completeness for  $\models_{\lambda K, I}$  (modulo the technical distinction between his “proof assignments” and our “models”, which are images of proof assignments under partial homomorphisms) [Lau70].

### 4.3. Soundness and Completeness

We show that if  $\vdash a$ , then  $\models a$  by demonstrating that if  $\alpha$  is a closed testimony term, i.e.,  $\mathcal{A}(\alpha) = \emptyset$ , then  $\llbracket \alpha \rrbracket$  belongs to every belief. We prove this using the following lemmas about open terms.

**Lemma 4.9**

Let  $\mathcal{F}$  be a lambda-K model and  $\mathcal{G}$  a functionally closed belief over  $\mathcal{F}$ . Let  $\eta \in \text{Env}(\mathcal{G})$  be any environment such that  $\eta(x^a) \in \mathcal{G}^a$  for all  $x^a$ . Then,

$$\forall \alpha \in \mathbf{T}. \alpha \vdash a \implies \mathcal{F}, \llbracket \alpha \rrbracket \eta, \mathcal{G} \models a$$

**Proof:** Straightforward induction on the structure of  $\alpha$ . See [Plo80, Sta82c]. ■

**Lemma 4.10**

Let  $\mathcal{F}$  be a lambda- $K$  model and  $\pi$  a hereditary permutation over  $\mathcal{F}$ . Let  $\eta \in \text{Env}(\mathcal{G})$ . Then,

$$\forall \alpha \in \mathbf{T}. \pi(\llbracket \alpha \rrbracket \eta) = \llbracket \alpha \rrbracket (\pi\eta),$$

where  $(\pi\eta)$  is the environment defined by  $(\pi\eta)(x) = \pi(\eta(x))$ .

**Proof:** Straightforward induction on the structure of  $\alpha$ , using the definition of  $\pi$  to show that  $\pi(\alpha\beta) = (\pi\alpha)(\pi\beta)$ . See [Plo80, Sta82a] for some related discussion. ■

**Theorem 4.11**

Let  $\mathcal{F}$  be a lambda- $K$  model and  $\mathcal{G}$  a functionally closed or invariant belief over  $\mathcal{F}$ . For any closed  $\alpha \in \mathbf{T}$  with  $\alpha \vdash a$ , we have

$$\mathcal{F}, \llbracket \alpha \rrbracket, \mathcal{G} \models a$$

**Proof:** From the two previous lemmas. ■

**Corollary 4.12**

$\forall a \in \mathbf{P}. \vdash a \implies \models_{P,Q} a$ , where  $P$  is either  $S$ ,  $\lambda K$ , or  $F\lambda K$ , and  $Q$  is either  $F$ ,  $C$ , or  $I$ . ■

The converse, *completeness*, for  $\models_{F\lambda K,S}$ , relies on the following lemma about term models.

**Lemma 4.13**

[Sta82b, Sta82c] Let  $\alpha$  be a testimony term with  $\alpha \vdash a$ . In any term model  $\mathcal{T}^\mathcal{E}$ , the following are equivalent:

1.  $\exists \beta. \alpha =_\mathcal{E} \beta \wedge \mathcal{A}(\beta) = \emptyset$
2. For all functionally closed beliefs  $\mathcal{G}$  over  $\mathcal{T}^\mathcal{E}$ ,  $[\alpha] \in \mathcal{G}_a$  ■

The proof of Lemma 4.13 given in [Sta82b] relies on the assumption that there is only one primitive proposition (propositional variable). The more general lemma stated here may be proved using an argument similar to the completeness proof for type inference given in [Hin83].

**Theorem 4.14**

$$\models_{\lambda K,F} a \iff \models_{F\lambda K,F} a \iff \vdash a$$

**Proof:** Show that  $\models_{\lambda K,F} a \implies \models_{F\lambda K,F} a \implies \vdash a \implies \models_{\lambda K,F} a$ . The first implication is trivial. For the second, assume that  $\models_{F\lambda K,F} a$ . In particular,  $\mathcal{T}_{\beta\eta} \models_{F\lambda K} a$ ; equivalently, for all functionally closed beliefs  $\mathcal{G}$  over  $\mathcal{T}_{\beta\eta}$ , we have  $\mathcal{T}_{\beta\eta}, \mathcal{G} \models a$ . So, by Lemma 4.13,  $\vdash a$ . The third implication is Corollary 4.12. ■

The completeness of  $\vdash$  for  $\models_{F\lambda K,F}$  is rather weak since we have used the term model instead of the standard model. Completeness for  $\models_{F\lambda K,F}$  is not vacuous, however, because it holds for the *fully inhabited* lambda- $K$  models. Allowing small lambda- $K$  models leaves open the question of whether functions in all beliefs over larger models prove additional theorems. A proof of completeness for  $\models_{S,F}$  would be much more satisfying, but might involve strengthening the difficult theorem of [Plo80].<sup>2</sup> Läuchli [Lau70] essentially proves completeness for  $\models_{\lambda K,I}$  using a correspondence between evidence models and Kripke models. Based on a result similar to Lemma 4.13 proved in [Sta82a], we conjecture that there is a direct proof of completeness for  $\models_{F\lambda K,I}$ .

<sup>2</sup>Essentially, Plotkin's lambda-definability theorem implies completeness for the standard model when each belief is replaced by a family of beliefs indexed by "possible worlds" of Kripke model.

## 5. Evidence Semantics for Relevant Implication

Our basic semantic idea for relevance logics is to restrict the functions in the beliefs of intuitionistic logic so that each function must always use its argument in some way. Certainly, constant functions must be ruled out, since they lead to the forbidden theorem  $a \rightarrow b \rightarrow a$ , and subsequent irrelevant consequences. Nonconstancy is not enough: because a nonconstant function may be constant over a large part of its domain, the nonconstant functions are not closed under composition. In general, we augment the structure of models or beliefs to include a binary relation on evidence, and require that every function in a belief preserve that relation. Many variations may be produced by choosing different sorts of relations, by choosing to preserve only positive instances of the relations, or negative ones as well, and by choosing to incorporate the relations in beliefs, where each function must preserve many different relations, or in models, where each function need preserve only one relation to qualify as evidence. We explore two sets of choices, differing in several of these parameters. Although we do not claim to give the definitive treatment of relevance, and are not even sure that there is a unique one, we believe that our analysis through relation-preserving functions gives a useful basis for understanding the semantics of relevance logics intuitively, and for comparing different logics.

### 5.1. Relevance Semantics Using Models Containing Independence-Preserving Functions

Our first semantics for relevance defines relevant functions to be those that preserve *independence* of evidence, and uses these independence relations as components of models. This approach is appropriate for  $\mathbf{RR}_\rightarrow$ . The intuition behind relevant functions in  $a \rightarrow b$  is that completely different evidence for  $a$ , given as input, must always yield completely different evidence for  $b$  as output. Perhaps the most obvious candidates for relevant functions are the *injective*, or *one-to-one*, functions. A closer inspection shows that injectivity is not enough. If  $\beta$  is injective, and  $\alpha_1 \neq \alpha_2$ , then  $\beta(\alpha_1) \neq \beta(\alpha_2)$ . But, suppose that  $\beta(\alpha_1)$  and  $\beta(\alpha_2)$  are of type  $a \rightarrow b$ , and that they differ at only one point in the domain associated with  $a$ . Such functions could hardly be taken as giving completely different evidence that  $a$  implies  $b$ . Even the stronger requirement that  $\forall \gamma. \beta(\alpha_1)(\gamma) \neq \beta(\alpha_2)(\gamma)$  is not enough, since it may be satisfied by merely permuting the range of the function  $\beta$ . Use of this pointwise definition of independent functions yields a logic, not explored here, in which an assumption may be used only once, so  $(a \rightarrow a \rightarrow b) \rightarrow a \rightarrow b$  fails to hold.

In order to guarantee that two pieces of functional evidence are completely different, we need to require that their ranges are completely different. For functions whose ranges contain structureless, independent, primitive objects, rather than other functions, the appropriate requirement is  $\forall \gamma, \delta. \beta(\alpha)(\gamma) \neq \beta(\alpha_2)(\delta)$ , that is, the ranges of  $\beta(\alpha_1)$  and  $\beta(\alpha_2)$  are disjoint. In general, the appropriate definition of a *relevant function* is that it preserves a *relevant independence* relation, as defined inductively below.

#### Definition 5.1

A *relevant independence relation* is a hierarchy  $\mathcal{R}$  of binary relations  $\mathcal{R}_a \subseteq \mathcal{U}_a \times \mathcal{U}_a$  such that:

1.  $\forall a \in \mathbf{P}_0. \alpha \mathcal{R}_a \beta \implies \alpha \neq \beta$
2.  $\forall a, b \in \mathbf{P}, \beta_1, \beta_2 \in \mathcal{U}_{a \rightarrow b}. (\beta_1 \mathcal{R}_{(a \rightarrow b)} \beta_2 \iff \forall \alpha_1, \alpha_2 \in \mathcal{U}_a. \beta(\alpha_1) \mathcal{R}_b \beta(\alpha_2))$  ■

Technically, it seems to make no difference whether or not  $\mathcal{R}_a$  is required to be symmetric, as well as irreflexive.

To develop some intuition for relevant independence and relevant functions, consider the case where, for  $a \in \mathbf{P}_0$ ,  $\alpha_1 \mathcal{R}_a \alpha_2 \iff \alpha_1 \neq \alpha_2$ . In this case, two functions are relevant independent if and only if their *primitive ranges* (the sets of primitive testimony generated by applying the functions to all possible sequences of arguments) are disjoint. Relevant functions might also be called *hereditarily injective* functions. Let  $a_1, \dots, a_n, b \in \mathbf{P}_0$ . A function  $\gamma \in \mathcal{U}_{a_1 \rightarrow \dots \rightarrow a_n \rightarrow b}$  preserves such an  $\mathcal{R}$  iff

$$\begin{aligned} & \forall \langle \alpha_1, \dots, \alpha_n \rangle, \langle \beta_1, \dots, \beta_n \rangle \in \mathcal{U}_{a_1} \times \dots \times \mathcal{U}_{a_n}. \\ & \langle \alpha_1, \dots, \alpha_n \rangle \neq \langle \beta_1, \dots, \beta_n \rangle \implies \gamma(\alpha_1) \dots (\alpha_n) \neq \gamma(\beta_1) \dots (\beta_n) \end{aligned}$$

In other words, such a relevant function  $\gamma$  is injective from  $\mathcal{U}_{a_1} \times \cdots \times \mathcal{U}_{a_n}$  to  $\mathcal{U}_b$ . In general, when the  $a_i$ s may not be primitive, then  $\gamma$  preserves  $\mathcal{R}$  whenever

$$\begin{aligned} \forall \langle \alpha_1, \dots, \alpha_n \rangle, \langle \beta_1, \dots, \beta_n \rangle \in \mathcal{U}_{a_1} \times \cdots \times \mathcal{U}_{a_n}. \\ (\exists i \in [1, n]. \alpha_i \mathcal{R}_{a_i} \beta_i) \implies \gamma(\alpha_1) \cdots (\alpha_n) \neq \gamma(\beta_1) \cdots (\beta_n) \end{aligned}$$

a condition somewhat weaker than injectivity. The converse does *not* hold. A function  $\gamma \in \mathcal{U}_a \rightarrow \mathcal{U}_b \rightarrow \mathcal{U}_c$  may be relevant, yet  $\gamma(\alpha_1)(\beta_1) = \gamma(\alpha_2)(\beta_2)$ , where  $\alpha_1 \mathcal{R}_a \alpha_2$ , and  $\beta_1 \neq \beta_2$ , but  $\neg \beta_1 \mathcal{R}_b \beta_2$ .

Models for this relevance semantics are the same as those for intuitionistic semantics, but with an independence relation that all functions must preserve.

### Definition 5.2

A *relational frame* is a pair  $\langle \mathcal{F}, \mathcal{R} \rangle$ , where  $\mathcal{F}$  is a frame, and  $\mathcal{R}$  is hierarchy of binary relations  $\mathcal{R}_a \in \mathcal{U}_a \times \mathcal{U}_a$  for each  $a \in \mathbf{P}$

An *independence model* is a relational frame  $\langle \mathcal{F}, \mathcal{R} \rangle$ , such that  $\mathcal{R}$  is a relevant independence relation, and

$$\forall a, b \in \mathbf{P}, \beta \in \mathcal{F}_{a \rightarrow b}, \alpha_1, \alpha_2 \in \mathcal{U}_a. \alpha_1 \mathcal{R}_a \alpha_2 \implies \beta(\alpha_1) \mathcal{R}_b \beta(\alpha_2)$$

A *standard independence model* is a  $\langle \mathcal{F}, \mathcal{R} \rangle$  with

$$\forall a, b \in \mathbf{P}. \mathcal{F}_{a \rightarrow b} = \{ \beta \in \mathcal{U}_{a \rightarrow b} \mid \forall \alpha_1, \alpha_2 \in \mathcal{U}_a. \alpha_1 \mathcal{R}_a \alpha_2 \implies \beta(\alpha_1) \mathcal{R}_b \beta(\alpha_2) \}$$

A *belief* over  $\langle \mathcal{F}, \mathcal{R} \rangle$  is just a belief over  $\mathcal{F}$

*Standard intuitionistic, full intuitionistic, functionally closed, classical, and invariant* beliefs may be considered over relational frames, just as over frames.

A belief is *independence-invariant* if it contains all of the functions in  $\mathcal{F}$  that are invariant under all hereditary permutations that preserve  $\mathcal{R}$

$\langle \mathcal{F}, \mathcal{R} \rangle, \mathcal{G}, \alpha \models a$  if  $a \in \mathcal{G}_a$

The subscripts  $F, C, T, I$  on  $\models$  refer to classes of beliefs, as before

Similarly,  $RI$  refers to the independence-invariant beliefs, and  $R$  to the independence models ■

The soundness of  $\vdash_{RR}$  with respect to various  $\models_Q$  follows from a slightly stronger result dealing with the behavior of relevant variables.

### Theorem 5.3

For all independence models  $\langle \mathcal{F}, \mathcal{R} \rangle$ ,  $\eta \in \text{Env}(\mathcal{F})$ ,  $a \in \mathbf{P}$ ,  $\alpha \vdash a$ ,  $x^b \in \mathbb{V}^b$ , and  $\beta, \gamma \in \mathcal{F}_b$ , the following two properties hold:

1.  $\llbracket \alpha \rrbracket \eta \in \mathcal{F}_a$
2.  $x^b \in \mathcal{R}_{\{y_1^{b_1}, \dots, y_n^{b_n}\}}(\alpha) \wedge \beta \mathcal{R}_b \gamma \wedge \delta_1, \epsilon_1 \in \mathcal{F}_{b_1} \wedge \cdots \wedge \delta_n, \epsilon_n \in \mathcal{F}_{b_n}$   
 $\implies \llbracket \alpha \rrbracket \eta[\beta/x, \delta_1/y_1, \dots, \delta_n/y_n] \mathcal{R}_a \llbracket \alpha \rrbracket \eta[\gamma/x, \epsilon_1/y_1, \dots, \epsilon_n/y_n]$

*Proof sketch:* By induction on the structure of  $\alpha$ . The steps are tedious but straightforward. ■

### Corollary 5.4

$\vdash_{RR} a \implies \models_{P,Q} a$ , where  $P$  is either  $S$  or  $R$ , and  $Q$  is either  $F, C, I$ , or  $RI$  ■

We conjecture that  $\mathbf{RR}_{\rightarrow}$  is complete for  $\models_{S,I}$ , and possibly for  $\models_{S,RI}$ . No completeness results for  $\mathbf{RR}_{\rightarrow}$  have been proved yet.

In responding to a very perceptive question from Mitchell Wand, we found an apparent paradox in the association of relevant functions with the lambda- $I$  calculus. The lambda- $I$  calculus is sufficiently powerful to compute many arithmetic functions, including addition and multiplication, using an encoding of positive integers into lambda-terms. But, addition and multiplication are not one-to-one, since, for example,  $1 + 2 = 2 + 1$ . There is really no paradox here, because the representations of integers in the lambda- $I$  calculus do not denote independent functions, so there are no particular restrictions on the behavior of relevant functions applied to integers.

### 5.2. Relevance Semantics Using Beliefs Containing Monotone Functions

The second semantic treatment of relevance is appropriate for the theory  $\mathbb{R}_{\rightarrow}$ . It is based on the idea that a relevant function  $\beta$  is one such that, if  $\alpha_2$  is strictly stronger evidence than  $\alpha_1$ , then  $\beta(\alpha_2)$  is strictly stronger than  $\beta(\alpha_1)$ . In this case, the ordering of evidence is incorporated into *beliefs*, rather than models.

#### Definition 5.5

A *lambda-I* model is a frame  $\mathcal{F}$  such that for all  $a, b, c \in \mathbb{T}$ , there exist elements

$$\begin{aligned} I_a &\in \mathcal{F}_a \\ B_{a,b,c} &\in \mathcal{F}_{(b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} \\ C_{a,b,c} &\in \mathcal{F}_{(a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c} \\ S_{a,b,c} &\in \mathcal{F}_{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} \end{aligned}$$

such that

$$\begin{aligned} I_a(\alpha) &= \alpha \\ B_{a,b,c}(\alpha)(\beta)(\gamma) &= \alpha(\beta(\gamma)) \\ C_{a,b,c}(\alpha)(\beta)(\gamma) &= \alpha(\gamma)(\beta) \\ S_{a,b,c}(\alpha)(\beta)(\gamma) &= (\alpha\gamma)(\beta\gamma) \end{aligned} \quad \blacksquare$$

Equivalently, the *lambda-I* models are those that are closed under *lambda-I* definability [Bar84]. Note that every standard or *lambda-K* model is a *lambda-I* model.

#### Definition 5.6

Let  $\mathcal{G}$  be a frame. A *hereditary strict partial ordering* of  $\mathcal{G}$  is a family of binary relations  $<_a$  such that

$$\begin{aligned} \text{For } a \in \mathbb{P}_0, & \quad <_a \text{ is a strict partial ordering of } \mathcal{G}_a \\ \forall a, b \in \mathbb{P}, \beta_1, \beta_2 \in \mathcal{G}_{a \rightarrow b} & \quad \beta_1 <_{a \rightarrow b} \beta_2 \iff (\forall \alpha \in \mathcal{G}_a . \beta_1(\alpha) <_b \beta_2(\alpha)) \end{aligned}$$

A *monotone belief* over frame  $\mathcal{F}$  is a pair  $\langle \mathcal{G}, < \rangle$ , where  $\mathcal{G}$  is a belief over  $\mathcal{F}$ ,  $<$  is a hereditary strict partial ordering of  $\mathcal{G}$ , and for all  $\beta \in \mathcal{G}_{a \rightarrow b}$ ,  $\alpha_1, \alpha_2 \in \mathcal{G}_a$  we have  $\alpha_1 < \alpha_2 \implies \beta(\alpha_1) < \beta(\alpha_2)$

A monotone belief  $\langle \mathcal{G}, < \rangle$  over  $\mathcal{F}$  is *monotone functionally closed* if

$$\mathcal{G}_{a \rightarrow b} = \{ \beta \in \mathcal{F}_{a \rightarrow b} \mid \forall \alpha_1, \alpha_2 \in \mathcal{G}_a . \beta(\alpha_1) \in \mathcal{G}_b \wedge \alpha_1 <_a \alpha_2 \implies \beta(\alpha_1) <_b \beta(\alpha_2) \} \quad \blacksquare$$

See [Gan80] for related discussion. Note that  $<_{a \rightarrow b}$  is defined *pointwise* on functions in  $\mathcal{G}_{a \rightarrow b}$ , while  $\mathcal{R}_{a \rightarrow b}$  is defined *globally* over the whole domains of such functions. For partial ordering relations, the pointwise definition allows repeated use of assumptions, and the objection raised for pointwise definition of independence relations is not compelling.

We say that  $\alpha \in \mathcal{F}_a$  is *monotone relevant evidence for a* if  $\alpha \in \mathcal{G}_a$  for every monotone belief  $\langle \mathcal{G}, < \rangle$  over  $\mathcal{F}$ . Relevant satisfaction and validity are defined as for intuitionistic logic:

#### Definition 5.7

$\mathcal{F}, \alpha, \langle \mathcal{G}, < \rangle \models a$  if  $\alpha \in \mathcal{G}_a$   
 $\mathcal{F}, \alpha \models_M a$  if  $\mathcal{F}, \alpha, \langle \mathcal{G}, < \rangle \models a$  for every monotone functionally closed belief  $\langle \mathcal{G}, < \rangle$  over  $\mathcal{F}$   
 Additional subscripts may be used to indicate sets of models as before  
 $\models_{\mathcal{M}, \mathcal{M}} a$  ( $\models_{F\mathcal{M}, \mathcal{M}} a$ ) if  $\mathcal{F} \models_M a$  for every (fully inhabited) *lambda-I* model  $\mathcal{F}$  \(\blacksquare\)

### 5.3. Soundness and Completeness

#### Theorem 5.8

Let  $\mathcal{F}$  be a lambda- $I$  model,  $\langle \mathcal{G}, < \rangle$  a monotone functionally closed belief over  $\mathcal{F}$  and  $\eta \in \text{Env}(\mathcal{G})$  an environment with  $\eta(x^a) \in \mathcal{G}_a$  for all  $x^a$ . Then,

$$\forall \alpha \in \mathbf{T}. \alpha \vdash a \implies \mathcal{F}, \llbracket \alpha \rrbracket \eta, \langle \mathcal{G}, < \rangle \models a$$

**Proof:** Straightforward induction on the structure of  $\alpha$ ; see [Gan80]. ■

#### Corollary 5.9

$\forall a \in \mathbf{P}. \vdash a \implies \models_{P,Q} a$ , where  $P$  is either  $S$ ,  $\lambda$ , or  $F\lambda$ , and  $Q$  is either  $F$ ,  $M$ ,  $C$ , or  $I$ . ■

The converse, *completeness*, for  $\models_{F\lambda, M}$ , relies on the following lemma about term models. Let  $\mathcal{T}_{I-\beta\eta}$  be the term model  $\mathcal{T}_{\beta\eta}$  restricted to lambda- $I$  terms.

#### Lemma 5.10

Let  $\alpha$  be a testimony term with  $\alpha \vdash_R a$ . In the term model  $\mathcal{T}_{I-\beta\eta}$ , the following are equivalent:

1.  $\mathcal{A}(\alpha) = \emptyset$
2. For all monotone functionally closed beliefs  $\langle \mathcal{G}, < \rangle$  over  $\mathcal{T}_{I-\beta\eta}$ , we have  $[\alpha] \in \mathcal{G}_a$  ■

The proof of Lemma 5.10 is similar in outline to that of 4.13, but involves some additional complications. Completeness is proved as for intuitionistic implication.

#### Theorem 5.11

$$\models_{\lambda, M} a \iff \models_{F\lambda, M} a \iff \vdash_R a \quad \blacksquare$$

We believe that this completeness theorem can be strengthened in two ways. By essentially the same proof techniques, we conjecture that completeness for both  $\models_{F\lambda, M}$  and  $\models_{F\lambda, F}$  can be proved, but we have not checked all the details. Thus it seems that we can either allow more functions in the model (completeness for  $\models_{F\lambda, M}$ ), or remove the restriction of monotonicity (completeness for  $\models_{F\lambda, F}$ ). Of the two, completeness for  $\models_{F\lambda, M}$  is by far the more interesting, since it suggests that monotonicity distinguishes relevant evidence from intuitionistic evidence, a result similar to Helman's [Hel77a]. We conjecture that  $\vdash_R$  is complete for  $\models_{S, M}$  as well, although there is less technical evidence for this conjecture.

## 6. Negation and Other Connectives

The usual constructive type-theoretic interpretation of  $\neg a$  is  $a \longrightarrow \Lambda$ , where  $\Lambda$  is “absurdity” and  $\longrightarrow$  is interpreted intuitionistically. Absurdity is axiomatized by  $\Lambda \longrightarrow b$  for all  $b$ . Prawitz [Pra65] gives the two changes to this interpretation that are required for the fault-tolerant logic of this paper. First, as argued in the introduction, a derivation of  $\Lambda$  from  $a$  does not provide evidence against  $a$  if the derivation of  $\Lambda$  does not really use the assumption  $a$ . So, the implication in the definition of negation should be *relevant*. Second, a  $\Lambda$  that implies everything is too strong. Therefore the axiom  $\Lambda \longrightarrow b$  for absurdity is dropped. To avoid confusing this “relevant” absurdity with the usual intuitionistic notion of absurdity, we may replace the symbol  $\Lambda$  with a symbol  $\Sigma$ , representing great surprise, but not implying anything else. Thus,  $\neg a$  is interpreted as  $a \rightarrow \Sigma$  (with  $\rightarrow$  *relevant*), indicating evidence against  $a$  that will cause surprise if positive evidence for  $a$  is discovered. Surprises should be rare, but they do happen occasionally, so we do not have  $\Sigma \rightarrow b$  for all  $b$ . Notice that both changes are needed: if  $\neg a$  is interpreted as  $a \longrightarrow \Sigma$ , with the intuitionistic  $\longrightarrow$ , then  $(a \wedge \neg a) \longrightarrow \neg b$  would hold for all  $b$ , although positive statements would not necessarily follow. In summary, no new rules are required for relevant reasoning about negation, merely the extension of  $\mathbf{P}_0$  by

P0.  $\Sigma \in \mathbb{P}_0$

and the convention that  $\neg a$  is an abbreviation for  $a \rightarrow \Sigma$ .

All of the usual logical connectives, including conjunction, disjunction, and universal and existential quantification, have been given type-theoretic interpretations for intuitionistic logic. See [Ste72] for a thorough treatment or [How80] for a brief, readable account of the main ideas. Essentially, conjunction corresponds to Cartesian product, disjunction to marked union, and quantification to dependent functional types, where the type of one component, or of the value returned by a function, may depend on another component, or on the argument to a function. Completeness for second-order quantification follows from the results of [MM85].

The extension of these interpretations to relevance logics is not immediate, because there are several plausible choices of definitions for relevant independence and ordering of evidence over these structures. Each choice leads to a different theory. It is tempting to start by introducing multiple conjunctions and disjunctions into a combined theory, to test by experience the usefulness of each. If we take the realizability interpretation of relevance logic totally seriously, however, we should attach the responsibility for defining independence to the implication sign, not to conjunction and disjunction. The resulting system of multiple implications appears to be quite clumsy, so an appeal to semantic principles could be useful.

In the syntactic analysis of proofs involving conjunctions and disjunctions, the appropriate definition of relevant occurrence of a variable is unlikely to coincide perfectly with that of free variable, even for the monotone semantics. Notice also that conventional representation techniques, such as the representation of a pair  $\langle a, b \rangle$  by a function mapping 0 to  $a$  and 1 to  $b$ , may not respect a given definition of independence. Careful study is required to determine which representations should be preserved in relevance logics, and whether such preservation requires additional restrictions on proofs of conjunctions and disjunctions corresponding to relevance restrictions on their functional representations. It would be particularly helpful to settle on an interpretation of second-order universal quantification early in the study, since many other connectives may be defined from that one (see [Pra65, Ste72]).

## 7. Conclusion and Directions for Further Investigation

The most commonly used models for intuitionistic and relevance logics are Kripke models involving “possible worlds.” We have examined an alternative in which models comprise sets of “evidence” for each proposition. Evidence models seem likely to be useful for studying a variety of logics, and we have proved some completeness theorems for intuitionistic propositional implications and relevant propositional implications. Since we have used the standard model theory for typed lambda calculus, the completeness theorems also yield characterizations of  $\lambda K$ -definable and  $\lambda I$ -definable functions over certain models. The characterization of  $\lambda I$ -definable functions seems to be new, and interesting apart from the application to relevance logic.

The evidence models discussed in this paper vary according to two parameters: the class of functions included in the models, and the class of beliefs used in the definition of validity. To be precise, a formula  $a$  is valid with respect to a class  $\mathcal{M}$  of models and class  $Q$  of beliefs if, for every model  $\mathcal{F}$  from  $\mathcal{M}$ , there is evidence  $\alpha \in \mathcal{F}_a$  belonging to every belief  $\mathcal{G} \subset \mathcal{F}$  from class  $Q$ . Thus, to preserve completeness for a given set of formulas, varying the class  $\mathcal{M}$  of models must be balanced against changes in the class  $Q$  of beliefs. Ideally, we would like to work with “standard” models consisting of all functions over chosen ground sets and with classes of beliefs that have simple mathematical characterizations. In fact, we have only proved completeness theorems with respect to classes of models which include models constructed from open and closed lambda terms. Perhaps, by altering the classes of beliefs we consider, we may be able to achieve completeness for more natural classes of models. In evaluating various choices, it is important to see how easy it is to give counterexamples to intuitionistic and relevant implications.

While the technical results to date are far from definitive, and the range of choices is larger than we would like, we believe that evidence models provide a basis for discussing those choices that allows direct application of intuition and experience. Questions such as, “*how is the strength of functional evidence*

related to the strengths of the function's values at different points" and, "must relevant evidence always map dependent pieces of evidence to dependent ones, as well as independent to independent" are not easy, but they are susceptible to intuitive discussion. Our evidence semantics provide a way to determine the impact of such discussion on formal systems of proof.

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