SUMMARY OF
"ON AUMANN'S NOTION OF COMMON KNOWLEDGE --
AN ALTERNATIVE APPROACH"

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ABSTRACT

We provide a bayesian model of knowledge which is based on the of infinite recursion of beliefs. The framework allows a direct formalisation of statements such as "everyone knows that (everyone knows)^m that an event A has occurred" and "A is common knowledge". An important and novel feature of our approach is that we explicitly formalise the statement "the information partitions are common knowledge" when we demonstrate the equivalence of our approach to the seminal contribution of Professor Aumann. This allows us to circumvent the self-reference which exists in that definition. The main theorem of the paper shows that given an Aumann structure, there exists a structure in the infinite recursion such that an event is common knowledge in the sense of Aumann if and only if it is common knowledge in our sense.

SUMMARY

[What follows is only a summary and description of results in Tan and Werlang(1985). Due to potential conflicts with journals,we regret not being able to submit the complete paper.]

Intuitively, an event is common knowledge if everyone knows it, everyone knows that everyone knows it, ... , and so on.

The notion of common knowledge is central (although frequently unstated) in economic modelling and game theory. The maintained hypothesis in most models is that the structure of the model and the rationality of the agents in the model are common knowledge. Indeed, it would be difficult to imagine how one should proceed in the analysis of a game if the rules of the game and the rationality of the other players were not common knowledge. Even Harsanyi's analysis of an incomplete information game, for instance, converts such a game into a larger bayesian game which is implicitly assumed to be common knowledge (at least by other writers using his framework).

However, a formal definition of common knowledge was not available to economists until the seminal paper of Aumann (1976). Aumann's framework is given formally in Section 2. As far as we know, it remains the most concise and elegant treatment of common knowledge in the literature.

The purpose of this paper is to offer an alternative definition of common knowledge which is less elegant and more cumbersome than Aumann's original framework. The framework offered here is based on the infinite recursion of beliefs introduced by Armbruster and Böge(1979), Böge and Eisele(1979) and Mertens and Zamir(1985). The definition of common knowledge given here is of course implicit in the three earlier contributions, however, the purpose of this paper is to make it explicit and to prove an equivalence theorem between the two, mathematically quite different, approaches.

Let us motivate the need for another definition of common knowledge, which by our own admission is less parsimonious than the original definition offered by Professor Aumann. Our interest in this alternative definition stems from both theoretical and practical considerations.

The first feature of the new definition is that the infinite recursion of beliefs offers a

mathematical language which allows us to formalise the intuitive notion of common knowledge directly. As a result, the defintion which we give in Section 4, translates verbally into the intuitive notion given in the first paragraph of this paper. In contrast, Aumann's definition , which makes use of agents' information partitions and the meet of the information partitions, requires a moment of thought before its relation to the intuitive notion is transparent. The price we have to pay for the alternative definition is that we have to invoke the more complex structure of the infinite recursion.

A related feature of the alternative approach is that the language of the infinite recursion allows one to easily define levels of knowledge which are lower than common knowledge. For instance, a statement like "everyone knows (everyone knows)^m that the event has occurred" can be formalised directly, without further requiring that the event be common knowledge.

Aumann's framework, which relies on information partitions of individuals, contains a self-reference. One has to implicitly assume that the information partitions are common knowledge themselves before one is able to define an event being common knowledge. The self-reference occurs precisely during that moment of thought when the reader is converting Aumann's mathematical definition into the intuitive notion of common knowledge (see the discussion in Section 2).

The advantage of the infinite recursion is that the framework is well defined mathematically, and since the notion of common knowledge is directly formalised, it does not contain self-references. Professor Aumann was of course aware of this issue in his original contribution and suggested that expanding the underlying uncertainty space would circumvent this problem. The results reported here, as well as related work by Brandenburg and Dekel (1985), demonstrates that one has to begin with the infinite recursion to achieve this.

For the readers familiar with the Aumann definition, an important feature of our approach (and to our knowledge, a novel feature in the literature) is that in Section 5, we explicitly formalise the assumption "the private information partitions are common knowledge", which is implicit and self-referential in the original definition.

Our interest in this alternative definition of common knowledge has its beginnings in our efforts to formally define the statement "rationality is common knowledge in a game" and to derive the behavioural implications of such a hypothesis (see Tan and Werlang, 1985). We found that it was not always convenient to work with Aumann's framework when the events of interest were rather complex, like rationality or a class of games. For example, to apply Aumann's framework for rationality, the space of uncertainty would have to be the set of all possible behaviour in a game, in which rationality is an element. Furthermore, one would have to find information partitions for the agents in order to define the event rationality as being common knowledge. We found these two steps unintuitive and thought that the first step of defining the space of all behaviour would be mathematically intractible. Furthermore, we were also interested in investigating the implications of the limited knowledge of rationality, a task which Aumann's definition did not facilitate.

In our search for a framework more suitable for the questions which we were addressing, we found that once one has invested the fixed capital of understanding the infinite recursion, many of the questions became simple exercises in that framework and the proofs of theorems were simple inductive arguments once the correct definitions were obtained. Hence, the definitions of "knowledge up to level m" and "common knowledge" that we give below are ones which we found extremely useful in other contexts. They provide an algorithm for anyone who is interest in defining any given event as being common knowledge, without having to think about the appropriate information partitions. Moreover, it also facilitates the investigation of what such a hypothesis might imply.

The main result of this paper is the equivalence theorem in Section 5 between the definition given here and the original definition of Aumann. What we show is that given an Aumann structure (defined formally in Section 2), there is a structure in terms of the infinite recursion of beliefs such that an event is common knowledge in the sense of Aumann if and only if the event is common knowledge in the sense given in Section 4. A related result by Brandenburg and Dekel (1985) complements our result neatly. They show that given an infinite recursion structure, there is an Aumann structure (i.e. a apace of uncertainty and information partitions and an element of the space) such that an event is common knowledge in the infinite recursion sence if and only if it is common knowledge int he sense of Aumann. Hence, these two results demonstrate that the two approaches are entirely equivalent and the choice of one or the other depends on the problem of interest.

In Aumann (1976), there was a discussion of the notion of reachability which he suggested as a way of directly capturing the intuitive notion of common knowledge. In the proof of the main theorem in Section 5, we formalise and make extensive use of reachability. However, it should be clear from a reading of the proof that the notion is far from intuitive and somewhat more cumbersome than the infinite recursion.

Section 2 gives Aumann's definition and provides a more technical discussion of some of the points raised above. A simple example, which is used throughout the paper, is provided there. Section 3 provides a introduction to the infinite recursion of beliefs, the bayesian interpretation of the framework, and the main result in that area. Section 4 provides the alternative definition of common knowledge and finally Section 5 states and proves the main equivalience result of this paper.

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