Counterfactuals and updates as inverse modalities

(Preliminary version)

Mark Ryan
School of Computer Science
University of Birmingham
Birmingham B15 2TT, UK.
mdr@cs.bham.ac.uk
http://www.cs.bham.ac.uk/~mdr

Pierre-Yves Schobbens
Institut d'Informatique
Facultés Universitaires de Namur
Rue Grandgagnage 21
5000 Namur, Belgium
pys@info.fundp.ac.be
http://www.info.fundp.ac.be/~pys

Odinaldo Rodrigues
Department of Computing
Imperial College
London, SW7 2BZ, UK.
otr@doc.ic.ac.uk
http://theory.doc.ic.ac.uk/~otr

Abstract

We point out a simple but hitherto ignored link between the theory of updates and counterfactuals and classical modal logic: update is a classical existential modality, counterfactual is a classical universal modality, and the link between the two (called the Ramsey rule) is simply the link between two inverse accessibility relations of a classical Kripke model.

1 Introduction

Background. An intuitive connection between theory change and counterfactuals was observed by F. P. Ramsey [17], who proposed what has become known as the Ramsey Rule:

> To find out whether the counterfactual 'if A were true, then B would be true' is satisfied in a state S, change the state S minimally to include A, and test whether B is satisfied in the resulting state.[1]

It was initially hoped that the AGM theory of belief revision [3, 13] would provide the right notion of minimal change. However, the intuitively acceptable AGM postulates for belief revision are known to be incompatible with the Ramsey Rule [2, 3].

It turns out that the theory of updates proposed by Katsuno and Mendelzon [10] is compatible with the Ramsey Rule [6]. Updates, like revisions, are a formalisation of theory change; but whereas revisions are intended to model changing knowledge about a fixed world, updates are intended to model a changing world. The difference between the formalisations of updates and revisions can be seen in terms of postulates; for example, the AGM postulate

$$A * B = A \wedge B \quad \text{if } A \wedge B \text{ is consistent}$$

[1]Actually, Ramsey proposed the rule only for non-counterfactual conditionals, but the term 'Ramsey Rule' is now taken to refer to counterfactuals too.

is accepted for revisions, but rejected for updates. The difference can also be seen in terms of operations on models; in revision, we measure the distance to the models of the old theory as a whole, while in update we measure the distance to them pointwise. Justifications of these differences (and further details and examples) can be found in [11, 6].

Our contribution. In this paper, we show that the standard treatments of updates (eg. [10]) and conditionals [18, 12, 14] are systems of multi-modal logic, whose Kripke accessibility relations are inverses of each other. This is the semantic equivalent of the Ramsey rule. For many of the standard postulates for updates and counterfactuals, we work out the correspondence property of the accessibility relation. This enables us to translate between postulates for counterfactuals and postulates for updates. In this way, we use Ramsey's Rule to translate between theories of update and theories of counterfactuals.

Structure. The paper is arranged as follows. Section 2 contains some preliminaries. In section 3, we show that updates and conditionals are systems of multi-modal logic, and that they have inverse accessibility relations. In section 4, we show that this is equivalent to the Ramsey rule, and in section 5 we translate the standard axioms for update into conditional axioms, and vice versa. Conclusions are in section 6.

2 Preliminaries

2.1 Multi-modal logic

We assume a propositional language L with finitely[2] many atomic propositions $p, q, r, \ldots$ and connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \Box, \Diamond$. The connectives $\Box$ and $\Diamond$ take two arguments; if A, B are formulas then so are $\Box_A B$ and $\Diamond_A B$. The set L is the set of atomic formulas $p, q, r, \ldots$; the set $\mathbf{L}$ is the set of all formulas over L.

The semantics of multi-modal logic is given as follows (cf. [5, 15, 16, 8, 7]). A *model* $M = \langle W, R, V\rangle$ of the multi-modal language L is a set W of *worlds*, an *accessibility relation* $R \subseteq \mathcal{P}(W) \times W \times W$ and a *valuation* $V : L \rightarrow \mathcal{P}(W)$. The ternary relation R may also be thought of as an $\mathcal{P}(W)$-indexed family $\{R_S \mid S \subseteq W\}$ of binary relations in $W \times W$.

The relation $\Vdash$ of *satisfaction* between a model $M = \langle W, R, V\rangle$, a world $x \in W$ and a formula A is defined inductively on A as follows.

$$\begin{array}{rll} x \Vdash_M p & \text{iff} & x \in V(p) \\ x \Vdash_M \neg A & \text{iff} & x \nVdash_M A \\ x \Vdash_M A \wedge B & \text{iff} & x \Vdash_M A \text{ and } x \Vdash_M B \\ x \Vdash_M \Box_A B & \text{iff} & \text{for each } y \in W,\ R_{|A|}(x,y) \text{ implies } y \Vdash_M B \\ x \Vdash_M \Diamond_A B & \text{iff} & \text{there is a } y \in W \text{ such that } R_{|A|}(x,y) \text{ and } y \Vdash_M B \end{array}$$

The missing connectives $\vee$, $\rightarrow$, $\leftrightarrow$ are defined by similar (standard) clauses.

In the context of a model M, $|A|$ is defined to be $\{x \in W \mid x \Vdash_M A\}$. Note that $|A \wedge C| = |A| \cap |C|$. We will use this fact in some proofs. The subscript on $\Vdash_M$ will usually be dropped in order to make the notation lighter.

The model M satisfies the formula A, written $M \Vdash A$, if $x \Vdash_M A$ for each $x \in W$. A *frame* $F = \langle W, R\rangle$ consists of a set of worlds and an accessibility relation. Such a frame

[2] The restriction that the number of propositional atoms be finite is imposed by Katsuno and Mendelzon, whose results we use.

F satisfies A, written $F \Vdash A$, if for each valuation V, we have $\langle W, R, V\rangle \Vdash A$. A formula A is *valid*, written $\models A$, if it is satisfied by every frame. A formula A is *satisfiable* in a model M if $|A| \neq \emptyset$. If $A_1, A_2, \ldots, A_n, B$ are formulas, the rule

$$\frac{A_1 \quad A_2 \quad \ldots \quad A_n}{B}$$

means: if each of the A_i is valid, then B is valid. Notice that this is rather weaker than asserting the axiom $A_1 \wedge \ldots \wedge A_n \rightarrow B$. The double-barred rule

$$\frac{A}{\overline{B}}$$

means: A is valid iff B is valid.

2.2 Inverse modalities

Classical modal logic is based on Kripke accessibility relations; it is thus natural to examine the logical counterparts of operations on relations. For instance, dynamic logic [16, 8, 7] (a logic of programs) uses relation composition (to express sequencing), transitive closure (to express iteration), union (to express non-deterministic choice). In this paper, we will be interested in the *inverse* operation on relations:

$$R^{-1}(x, y) = R(y, x)$$

Given a unary modality $\Box$ associated with accessibility relation R, we will use $\bar{\Box}$ to denote the modality associated with R^{-1}.

Inverse modalities have already been used in modal logics: in linear temporal logic, they are called *past* modalities. The table below summarises their intuitive meaning. These inverse modalities should not be confused with the dual modalities, nor with the inverse of the dual (which is of course dual of the inverse).

modality $\Box B$	inverse $\bar{\Box} B$	dual $\Diamond B$	inverse dual $\bar{\Diamond} B$
henceforth B	up to now, B	eventually B	once upon a time, B
tomorrow B	yesterday B	tomorrow B	yesterday B
I believe that B	in all situations where my beliefs admit the current situation, B is true	B is consistent with my beliefs	the current situation is consistent with my beliefs, and B
necessarily B	if reality is possible, B	possibly B	reality is possible, and B
Any result of program P satisfies B	Any input of program P satisfies B	Some result of program P satisfies B	Some input of program P satisfies B

In order to understand the reading of $\bar{\Box}$ for a particular reading of $\Box$, one should think about the meaning of the accessibility relation R, and then about its inverse. For example, if $\Box B$ is 'I believe that B', then $R(x, y)$ means: if the actual world is x, then y is a possible world according to my beliefs (in x). Thus, $\Box B$ holds at x if B holds in all worlds which could be the actual world according to my beliefs.

Now look at the inverses. $R^{-1}(x, y)$ means: if the actual world is y, then x is a possible world according to my beliefs (in y). So, $\bar{\Box}B$ holds at x if B holds in all worlds which, if it is the actual world then x is a possible world according to my beliefs in it. Thus, $\bar{\Box}B$ says: if the actual world is consistent with my beliefs, then B.

2.2.1 Axiomatising inverse modalities

How can we axiomatise the link between a modality and its inverse? It turns out that there are two ways, depending on the language we wish to use: We may either wish to keep the positive and negative modalities in separate languages, or to have a single language including both.

We first look at the latter:

Theorem 2.1 The following two axioms (each of which is also given in its dual form), when added to the classical rules of distribution and necessitation, axiomatise a pair of inverse modalities.

$$\begin{array}{lll} (1) & B \to \Box\bar{\Diamond}B & \Diamond\bar{\Box}B \to B \\ (2) & B \to \bar{\Box}\Diamond B & \bar{\Diamond}\Box B \to B \end{array}$$

Proof Let $\langle W, (S, R)\rangle$ be a frame, where R is the accessibility relation for $\Box, \Diamond$, and S is the relation for $\bar{\Box}, \bar{\Diamond}$. We show that the axioms hold in the frame iff $S = R^{-1}$.

($\Leftarrow$) is straightforward. For ($\Rightarrow$), suppose $S(x, y)$; choose the valuation V s.t. $V(p) = \{x\}$ then $x \Vdash p$, so by (2) $x \Vdash \bar{\Box}\Diamond p$ so $y \Vdash \Diamond p$ so $\exists z, R(y, z) \wedge z \Vdash p$. But by $V, z = x$, so $R(y, x)$. So $S \subseteq R^{-1}$. The converse inclusion is similar, but uses (1). □

We might prefer a rule that allows us to work with two separate sublanguages, one containing only the modalities $\Box, \Diamond$ and the other the modalities $\bar{\Box}, \bar{\Diamond}$. Theorems in one logic could be translated in the other one, provided we find inference rules that do not mix languages.

Theorem 2.2 Axiom (1) is equivalent to the following rule (which is also given in its dual form):

$$\frac{B \to \bar{\Box}C}{\Diamond B \to C} \qquad \frac{\bar{\Diamond}C \to B}{C \to \Box B}$$

Axiom (2) is equivalent to the rule (also given in dual form):

$$\frac{\Diamond B \to C}{B \to \bar{\Box}C} \qquad \frac{C \to \Box B}{\bar{\Diamond}C \to B}$$

Proof First half:

($\Rightarrow$) $B \to \bar{\Box}C$ by hyp
$\Diamond B \to \Diamond\bar{\Box}C$ by K,MP
$\Diamond B \to C$ by (1) dual form

($\Leftarrow$) $\bar{\Diamond}B \to \bar{\Diamond}B$
$B \to \Box\bar{\Diamond}B$ by rule (dual form)

The other half is symmetrical. □

Thus, the rule

$$\frac{B \to \bar{\Box} C}{\Diamond B \to C}$$

completely axiomatises the relationship of inverse between $\Box$ and $\bar{\Box}$. We will see in section 3.2 that the multi-modal version of this rule,

$$\frac{B \to \bar{\Box}_A C}{\Diamond_A B \to C}$$

exactly expresses the Ramsey rule for counterfactuals. $\bar{\Box}_A C$ will be interpreted as 'if A were true, then C would be true; $\Diamond_A B$ will be interpreted as the update of B by A, so the rule states that the counterfactual 'if A, C' is supported in a state B iff the state obtained by updating B with A supports C.

2.3 Other preliminaries

If the formula A has no modalities, we write mod(A) to mean the set of valuations which make A true, in the usual propositional way. Notice the difference between mod and $|\ |$.

The formula A over L is *complete* if for all formulas B over L, $A \models B$ or $A \models \neg B$.

If $(X, \leq)$ is a pre-ordered set and $Y \subseteq X$, then $\text{Min}_{\leq}(Y)$ is the set of $\leq$-minimals in Y, i.e. $\text{Min}_{\leq}(Y) = \{y \in Y \mid \forall x \in Y. x \not< y\}$. An order $\leq$ on a set of worlds W in some model M is *stoppered* if for every non-empty $|A| \subseteq W$ and $y \in |A|$ there is $x \in \text{Min}_{\leq}(|A|)$ with $x \leq y$.

3 Updates and counterfactuals

We show that updates and conditionals are systems of modal logic, in sections 3.1 and 3.2 respectively. In section 3.3, we observe that they have inverse accessibility relations.

3.1 Updates

In [11], the difference between updates and revisions was pointed out and in [10] new postulates for updates were proposed. These postulates are similar to those for revisions [4] and are presented in Table 1. In the last column, we have indicated the name used in belief revision. As can be seen, properties of updates and revisions have much in common, explaining the historical confusion; indeed, in [9] updates are referred to as *pointwise revisions.*

Katsuno/Mendelzon's update axioms U8 and U4.1 suggest that update behaves like an existential modality on its second argument. Moreover, they observe [9, theorems 6.1, 6.3], [10, theorem 3.4] that

$$\text{mod}(q \Diamond p) = \bigcup_{y \in \text{mod}(q)} \text{Min}_{\leq_y}(\text{mod}(p))$$

where $\leq_y$ is a preorder of closeness to y ($x \leq_y z$ means that x is at least as close to the world y as z is). If we write $\Diamond_p q$ in place of $q \Diamond p$ and define the multi-modal model $M = \langle W, R, V\rangle$ where W is the set of valuations of the language, the relation R is given by

$$R_S(x, y) \Leftrightarrow x \in \text{Min}_{\leq_y}(S),$$

name [10]	axiom	name [4]
U1	$q \diamond p \to p$	K*2
U2.1	$q \to p$ implies $q \to q \diamond p$	
U2.2	$q \to p$ implies $q \diamond p \to q$	K*4w
U3	$q \diamond p$ satisfiable, if p, q satisfiable	$\sim$ K*5
U4.1	$q \leftrightarrow r$ implies $q \diamond p \leftrightarrow r \diamond p$	
U4.2	$q \leftrightarrow r$ implies $p \diamond q \leftrightarrow p \diamond r$	K*6
U5	$(q \diamond r) \wedge p \to q \diamond (r \wedge p)$	K*7
U6	$q \diamond p \to r, q \diamond r \to p$ imply $q \diamond p \leftrightarrow q \diamond r$	
U7	q complete implies $(q \diamond p) \wedge (q \diamond r) \to q \diamond (p \vee r)$	
U8	$(q \vee r) \diamond p \leftrightarrow (q \diamond p) \vee (r \diamond p)$	

Table 1: Update postulates according to Katsuno/Mendelzon [10], using their notation. They use $\diamond$ as an infix operator; $q \diamond p$ means q updated by p.

and $V(p) = \text{mod}\,(p)$, then Katsuno/Mendelzon's observation is equivalent to the standard satisfaction condition for an existential modality

$$x \Vdash \diamond_A B \quad \text{iff} \quad \text{there exists } y \text{ s.t. } R_{|A|}(x, y) \text{ and } y \Vdash B.$$

Adopting this suggestion, we recast U1-U8 as multi-modal axioms in Table 2.

To obtain the classical properties of a modality, we should also have necessitation:

$$\frac{B}{\Box_A B}$$

Theorem 3.1 Necessitation follows from the axioms and rules in Table 2.

Proof Assume $\models B$. Then $\models \neg B \leftrightarrow \bot$, and by U4.1: $\models \diamond_A \neg B \leftrightarrow \diamond_A \bot$. By U2.2: $\models \diamond_A \bot \to \bot$, so $\models \diamond_A \bot \leftrightarrow \bot$. Therefore, $\models \diamond_A \neg B \leftrightarrow \bot$, hence $\models \neg \diamond_A \neg B$. □

As usual within the framework of modal logic, we can study the 'correspondence properties' on R imposed by each of the axioms U1-U8. Moreover, since R is expressed in terms of $\leq$, we can look at what conditions of R and $\leq$ each of the conditions corresponds to. This occupies us for the remainder of this section.

Theorem 3.2 An axiom scheme or rule in Table 2 holds in a frame $F = \langle W, R \rangle$ iff R has the corresponding property stated in Table 3.

Compare correspondence theorems for standard modal logic, eg. [15, §5.2], [5, theorems 1.12, 1.13].

Proof The proofs follow the usual pattern in correspondence theory. In the $\Leftarrow$ direction, we add to the frame $\langle W, R \rangle$ an arbitrary valuation V to form the model $M = \langle W, R, V \rangle$, and show that the constraint on R is enough to guarantee that the axiom scheme or rule is satisfied at any point $x \in W$. In the $\Rightarrow$ direction, we make a judicious choice of the valuation and an instance of the scheme, to show that the constraint on R must hold.

name [10]	rewritten as
U1	$\Diamond_A B \to A$
U2.1	$\dfrac{B \to A}{B \to \Diamond_A B}$
U2.2	$\dfrac{B \to A}{\Diamond_A B \to B}$
U3	$\Diamond_A B$ satisfiable if A, B satisfiable
U4.1	$\dfrac{B \leftrightarrow C}{\Diamond_A B \leftrightarrow \Diamond_A C}$
U4.2	$\dfrac{B \leftrightarrow C}{\Diamond_B A \leftrightarrow \Diamond_C A}$
U5	$\Diamond_A B \wedge C \to \Diamond_{A \wedge C} B$
U6	$\dfrac{\Diamond_A B \to C \quad \Diamond_C B \to A}{\Diamond_A B \leftrightarrow \Diamond_C B}$
U7	B complete implies $\Diamond_A B \wedge \Diamond_C B \to \Diamond_{A \vee C} B$
U8	$\Diamond_A (B \vee C) \leftrightarrow \Diamond_A B \vee \Diamond_A C$

Table 2: Update postulates of Table 1 rewritten as modal logic axioms and rules

name	corresponding property of R	corresp. property of $\leq_y$
R1	$R_S(x, y)$ implies $x \in S$	–
R2.1	$y \in S$ implies $R_S(y, y)$	$y \leq_y x$ (weak centering)
R2.2	$y \in S$ and $R_S(x, y)$ imply $x = y$	$x \leq_y y$ implies $x = y$
R3	$S \neq \emptyset$ implies $\forall y \exists x . R_S(x, y)$	$\leq_y$ stoppered[a]
R5	$x \in S$ and $R_T(x, y)$ imply $R_{S \cap T}(x, y)$	–
R6	if $x \in S$, $\neg R_S(x, y)$ then $\exists z . R_S(z, y) \wedge$ $\forall T . (z \in T \to \neg R_T(x, y))$[a]	$\leq_y$ stoppered[a].
R7	$R_S \cap R_T \subseteq R_{S \cup T}$	

[a]sufficient condition only.

Table 3: Properties of R corresponding to the axioms/rules in Table 2

U1 $\Leftarrow$ R1. Let V be any valuation, and let $M = \langle W, R, V \rangle$. Suppose $x \Vdash_M \Diamond_A B$. Let $S = |A|$, and take a y such that $R_S(x,y)$ and $y \Vdash B$. By R1, $x \in S$, so $x \Vdash_M A$.

U1 $\Rightarrow$ R1. Suppose $R_S(x,y)$ in the frame $\langle W, R \rangle$; pick V such that $V(p) = S$ and $V(q) = \{y\}$. Since $y \Vdash q$, we have $x \Vdash \Diamond_p q$; so $x \Vdash p$ by U1, and $x \in S$ by def. of V.

U2.1 $\Leftarrow$ R2.1. Suppose $\models B \to A$ and $x \Vdash B$. Then $x \Vdash A$. We have $x \Vdash \Diamond_A B$ iff $\exists y$ s.t. $y \Vdash B$, and $R_{|A|}(x,y)$, which is true if we set $y = x$.

U2.1 $\Rightarrow$ R2.1. Suppose $y \in S$; we prove that $R_S(y,y)$. Let V be such that $V(p) = S$ and $V(q) = \{y\}$. It follows that $\models q \to p \vee q$, and therefore, by U2.1, $y \Vdash \Diamond_{p \vee q} q$. But $|p \vee q| = S$, so $\exists z \Vdash q$ s.t. $R_S(y,z)$. But $V(q) = \{y\}$. Thus, z must be y, and therefore $R_S(y,y)$.

U2.2 $\Leftarrow$ R2.2. Suppose $\models B \to A$ and $x \Vdash \Diamond_A B$. Take y such that $R_{|A|}(x,y)$ and $y \Vdash B$. Since $\models B \to A$, $y \in |A|$; so by R2.2, $x = y$; so $x \Vdash B$.

U2.2 $\Rightarrow$ R2.2. Suppose $y \in S$ and $R_S(x,y)$. Let V be such that $V(p) = S$ and $V(q) = \{y\}$. It follows that $\models q \to p \vee q$, and therefore, by U2.2, $\Diamond_{p \vee q} q \to q$. Since $y \Vdash q$, $x \Vdash \Diamond_{p \vee q} q$, so $x \Vdash q$, so $x = y$ (by def. of V).

U3 $\Leftarrow$ R3. Take any valuation. Suppose A and B are satisfiable in M, so take $a \Vdash A$ and $b \Vdash B$. Then $|A| \neq \emptyset$, so by R3 there is an x with $R_{|A|}(x,b)$. Hence, $x \Vdash \Diamond_A B$.

U3 $\Rightarrow$ R3. Suppose $S \neq \emptyset$ and y is given. Take V such that $V(p) = S$ and $V(q) = \{y\}$. Then by U3, $\Diamond_p q$ is satisfiable, i.e. there is an x with $x \Vdash \Diamond_p q$. Then there is a z with $R_S(x,z)$ and $z \Vdash q$. But by choice of V, $z = y$; so $R_S(x,y)$.

U5 $\Leftarrow$ R5. Suppose $x \Vdash \Diamond_A B \wedge C$. Since $x \Vdash \Diamond_A B$, $\exists y \Vdash B$, $R_{|A|}(x,y)$. By R5, $R_{|A| \cap |C|}(x,y)$ and thus $x \Vdash \Diamond_{A \wedge C} B$.

U5 $\Rightarrow$ R5. Suppose $x \in S$ and $R_T(x,y)$. Pick V such that $V(p) = S$, $V(q) = \{y\}$, and $V(r) = T$. Then $y \Vdash q$ and $R_T(x,y)$ imply that $x \Vdash \Diamond_r q$. Since $x \Vdash p$, by U5 $x \Vdash \Diamond_{r \wedge p} q$. Therefore, $\exists y'$ s.t $y' \Vdash q$ and $R_{S \cap T}(x,y')$. But y' must be y, since $V(q) = \{y\}$, so $R_{S \cap T}(x,y)$.

U6 $\Leftarrow$ R6. The condition R6 deserves an explanation. Intuitively, it says: if x is eliminated from our preferred set, it is justified by some z that consistently gets preferred.

Suppose $\models \Diamond_A B \to C$, $\models \Diamond_C B \to A$, and $x \Vdash \Diamond_A B$. Then $\exists y_0\, R_{|A|}(x,y_0)$ and $y_0 \Vdash B$; and from $\models \Diamond_A B \to C$ we have $x \Vdash C$. Suppose for a contradiction that $x \nVdash \Diamond_C B$; then $\neg R_{|C|}(x,y_0)$. Now apply R6 with $S = |C|$, and take $T = |A|$; there exists z with $R_{|C|}(z,y_0) \wedge z \nVdash A$. Use $\models \Diamond_C B \to A$ to show that $z \nVdash \Diamond_C B$, which contradicts $R_{|C|}(z,y_0) \wedge y_0 \Vdash B$.

U7 $\Leftarrow$ R7 Suppose $x \Vdash \Diamond_A B \wedge \Diamond_C B$. Then exist y_1, y_2 with $R_{|A|}(x,y_1)$, $R_{|C|}(x,y_2)$ and $y_1, y_2 \Vdash B$. Since B complete, $y_1 = y_2$, so $(R_{|A|} \cap R_{|C|})(x,y_1)$, hence by R7 $R_{|A| \cup |C|}(x,y_1)$

U7 $\Rightarrow$ R7 Supposing $R_S(x,y)$ and $R_T(x,y)$, pick V such that $V(p) = S$, $V(q) = \{y\}$ and $V(r) = T$. Then $y \Vdash q$, so $x \Vdash \Diamond_p q \wedge \Diamond_r q$. By U7, $x \Vdash \Diamond_{p \vee r} q$, so there exists z (which by definition of V must be y), with $R_{|p \vee r|}(x,z)$. But $|p \vee r| = S \cup T$, so $R_{S \cup T}(x,y)$. □

The axioms U4.1, U4.2 and U8 are simply the usual properties of an existential modality; thus, they do not constrain the accessibility relation.

Theorem 3.3 The axiom schemes U4.1, U4.2 and U8 hold in any frame.

Proof U4.1. Suppose $\models B \leftrightarrow C$ and $x \Vdash \diamond_A B$. Then there exists y with $R_{|A|}(x,y)$ and $y \Vdash B$. But also, $y \Vdash C$, so $x \Vdash \diamond_A C$. The other half is similar.

U4.2. Suppose $\models B \leftrightarrow C$ and $x \Vdash \diamond_B A$. Then there exists y with $R_{|B|}(x,y)$ and $y \Vdash A$. But also, $R_{|C|}(x,y)$, so $x \Vdash \diamond_C A$. The other half is similar.

U8. $x \Vdash \diamond_A(B \vee C)$ iff $\exists y\, R_{|A|}(x,y), y \Vdash B \vee C$ iff $\exists y_1\, R_{|A|}(x,y_1), y_1 \Vdash B$ or $\exists y_2\, R_{|A|}(x,y_2), y_2 \Vdash C$ iff $x \Vdash \diamond_A B \vee \diamond_A C$.

□

3.2 Counterfactuals

According to [18, 12, 14], the counterfactual 'if A was the case, then B would be the case' may be interpreted by: "In all closest worlds satisfying A, we find that B holds." It is well-known that counterfactuals have the properties of classical universal modalities [1, 12]. The counterfactual 'if A was the case, then B would be the case' holds at a world x if B holds in all y in $\mathrm{Min}_{\leq_x}|A|$. But this relation between x and y is simply the inverse of the relation R given in section 3.1. Thus, we see that counterfactuals are not just a universal modality; they are the inverse dual modality to updates. The counterfactual sentence 'if A was the case, then B would be the case' can be written $\bar{\Box}_A B$, where

$$x \Vdash \bar{\Box}_A B \quad \text{iff} \quad \text{for all } y,\ R^{-1}_{|A|}(x,y) \text{ implies } y \Vdash B.$$

One can perform the same analysis as we did for updates, namely, the correspondence theory for standard postulates for counterfactuals. This is work in progress.

4 Inter-translating systems for counterfactuals and updates

We have observed that postulates for updates correspond to particular properties of the accessibility relation R, and similarly for counterfactuals, whose postulates correspond to properties of the inverse relation R^{-1}. This means that a particular postulate for updates can be translated into a postulate for counterfactuals. The proof of the equivalence between the postulates can be performed

- either by going via the accessibility relation R;
- or by working directly with the axiomatisations of theorem 2.1;
- or by working with the Ramsey Rule (theorem 2.2).

We have worked out the counterfactual counterpart of the postulates U1-U8. Our preliminary findings at the time of going to press are summarised in the following table.

name	counterfactual axiom
C1	$\bar{\Box}_A A$
C2.1	$B \to A$ implies $B \to \bar{\Diamond}_A B$
C2.2	$B \to A$ implies $B \to \bar{\Box}_A B$
C3	$B \to \bar{\Box}_A \bot$ implies $\neg B$ or $\neg A$
C4.2	$B \leftrightarrow C$ implies $\bar{\Box}_B A \leftrightarrow \bar{\Box}_C A$
C5	$(\bar{\Box}_A C \wedge \bar{\Box}_B C) \to \bar{\Box}_{A \vee B} C$[a]
C6	$B \to \bar{\Box}_A C, B \to \bar{\Box}_C A$ imply $\bar{\Box}_A B \leftrightarrow \bar{\Box}_C B$
C7	B complete implies $\bar{\Diamond}_A B \wedge \bar{\Diamond}_C B \to \bar{\Diamond}_{A \vee C} B$

[a] using U1,U4,U5

For example, the translation of U1 is obtained as follows.

$$\models \Diamond_A B \to A \;\Leftrightarrow\; \models B \to \bar{\Box}_A A \quad \text{Ramsey Rule}$$
$$\Leftrightarrow\; \models \bar{\Box}_A A$$

5 Conclusions

The link between counterfactual and updates, often considered as esoteric, is only the usual link between a relation and its inverse; counterfactuals, as is known, can be considered as a universal modality, and update as its inverse existential modality.

Some work remains to be done:

- obtain similar results for the inverse modalities used in other logics (e.g. temporal logic)
- study also known axioms proposed for counterfactuals and derive the corresponding update axiom.

5.1 Acknowledgements

Thanks to the ModelAge'96 Workshop referees for their comments on an early draft of this paper, and to Esprit for partial funding under the WG ModelAge (8319).

References

[1] B. F. Chellas. Basic conditional logic. *Journal of Philosophical Logic*, 4:133–153, 1980.

[2] P. Gärdenfors. Belief revision and the Ramsey test for conditionals. *Philosophical Review*, 91:81–93, 1986.

[3] P. Gärdenfors. *Knowledge in Flux: Modelling the Dynamics of Epistemic States.* MIT Press, 1988.

[4] Peter Gärdenfors. *Knowledge in Flux: Modeling the Dynamics of Epistemic States.* MIT Press, Bradford Books, Cambridge, MA, 1988.

[5] R. Goldblatt. *Logics of Time and Computation.* CSLI Lecture Notes, 1987.

[6] G. Grahne. Updates and counterfactuals. In *Proc. Second International Conference on Principles of Knowledge Representation and Reasoning (KR '91)*, pages 269–276. Morgan Kaufmann, San Francisco, CA, 1991.

[7] D. Harel. Dynamic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Phylosophical Logic, Vol II*, page 715 ff. Reidel Publ. Co., Dordrecht, Holland, 1987.

[8] David Harel. *First-Order Dynamic Logic*, volume 68 of *Lecture Notes in Computer Science*. Springer-Verlag, New York, 1979.

[9] Hirofumi Katsuno and Alberto O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.

[10] Hirofumi Katsuno and Alberto O. Mendelzon. On the difference between updating a knowledge base and revising it. In Peter Gärdenfors, editor, *Belief Revision*, number 29 in Cambridge Tracts in Theoretical Computer Science, pages 183–203. Cambridge University Press, 1992.

[11] Arthur M. Keller and Marianne Winslett Wilkins. On the use of an extended relational model to handle changing incomplete information. *IEEE Transactions on Software Engineering*, 11(7):620–633, July 1985.

[12] David K. Lewis. *Counterfactuals*. Harvard University Press, 1973.

[13] D. Makinson. How to give it up: A survey of some formal aspects of the logic of theory change. *Synthèse*, 62:347–363, 1985.

[14] John L. Pollock. A refined theory of counterfactuals. *Journal of Philosophical Logic*, 10:239–266, 1981.

[15] S. Popkorn. *First Steps in Modal Logic*. Cambridge University Press, 1994.

[16] V. R. Pratt. Models of program logics. In *Proc. 20th IEEE Symp. on Foundations of Computer Science*, pages 115–122, 1979.

[17] F. P. Ramsey. Truth and probability. In R. B. Braithwaite, editor, *The Foundations of Probability and other Logical Essays*. Harcourt Brace, New York, 1931.

[18] Robert C. Stalnaker. *A Theory of Conditionals*, volume 2 of *American Philosophical Quarterly Monograph Series (ed. Nicholas Rescher)*, pages 98–112. Blackwell, Oxford, 1968.