

Resolving Distributed Knowledge

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Distributed knowledge is the sum of the knowledge in a group; what someone who is able to discern between two possible worlds whenever *any* member of the group can discern between them, would know. Sometimes distributed knowledge is referred to as the potential knowledge of a group, or the joint knowledge they could obtain if they had unlimited means of communication. In epistemic logic, the formula $D_G\varphi$ is intended to express the fact that group G has distributed knowledge of φ , that there is enough information in the group to infer φ . But this is not the same as reasoning about *what happens if the members of the group share their information*. In this paper we introduce an operator R_G , such that $R_G\varphi$ means that φ is true after G have shared all their information with each other – after G 's distributed knowledge has been *resolved*. The R_G operators are called *resolution operators*. Semantically, we say that an expression $R_G\varphi$ is true iff φ is true in what van Benthem [11, p. 249] calls (G 's) *communication core*; the model update obtained by removing links to states for members of G that are not linked by *all* members of G . We study logics with different combinations of resolution operators and operators for common and distributed knowledge. Of particular interest is the relationship between distributed and common knowledge. The main results are sound and complete axiomatizations.

1 Introduction

In epistemic logic [6, 8, 5] different notions of *group knowledge* describe different ways in which knowledge can be associated with a group. *Common knowledge* is stronger than individual knowledge: that something is common knowledge requires not only that everybody in the group knows it, but that everybody knows that everybody knows it, and so on. *Distributed knowledge*, on the other hand, is weaker than individual knowledge: distributed knowledge is knowledge that is distributed throughout the group even if no individual knows it.

More concrete informal descriptions of the concept of distributed knowledge abound, but they are often inaccurate descriptions of the concept as formalized in standard epistemic logic. A misconception is that something is distributed knowledge in a group if the agents in the group could get to know it after some (perhaps unlimited) communications between them¹. To see that this interpretation must be incorrect, consider the formula $D_{\{1,2\}}(p \wedge \neg K_1 p)$. In this formula, $D_G\varphi$ and $K_i\varphi$ mean that φ is distributed knowledge in the group G , and individual knowledge of agent i , respectively. Thus, the formula says that it is distributed knowledge among agents 1 and 2 that p is true and that agent 1 does not know p . This formula is *consistent* (also when we assume that knowledge has the S5 properties). However, it is not possible that agents 1 and 2 both can get to know that p is true and that agent 1

¹Some examples of informal descriptions of distributed knowledge from the literature include “A group has distributed knowledge of a fact φ if the knowledge of φ is distributed among its members, so that by pooling their knowledge together the members of the group can deduce φ ” [6]; “.. it should be possible for the members of the group to establish φ through communication” [12, 10]; “.. the knowledge that would result of the agents could somehow ‘combine’ their knowledge” [12]. These descriptions can at least give a reader the impression that distributed knowledge is about internal communication in the group of agents.

does not know that p is true (assuming the S5 properties of knowledge), no matter how much they communicate (or “pool” their knowledge). The “problem” here is that in a formula $D_G\psi$, ψ describes the possible states of the world as they were before any communication or other events took place, so a more accurate reading of $D_{\{1,2\}}(p \wedge \neg K_1 p)$ would perhaps be that it follows from the combination of 1 and 2’s knowledge that $p \wedge \neg K_1 p$ were true before any communication or other events took place. More technically, the “problem” is due to the standard compositional semantics of modal logic: in the evaluation of $D_G\phi$, the D_G operator picks out a number of states considered possible by the group G (actually the states considered possible by *all* members of the group), and then ϕ is evaluated in each of these states *in the original model, without any effect of the D_G operator*.

But we don’t really consider this a problem. There are other interpretations of distributed knowledge where the consistency of the mentioned formula makes perfect sense, such that distributed knowledge being the knowledge of a third party, someone “outside the system” who somehow has access to the epistemic states of the group members. It shows, however, that it does not make sense to interpret distributed knowledge as something that is true after the agents in the group have communicated *with each other* – with the standard semantics.

In this paper we introduce and study an alternative group modality R_G , where $R_G\phi$ means (roughly speaking) that ϕ is true after the agents in the group have shared all their information with each other. We call that *resolving* distributed knowledge, and the R_G operators are called *resolution operators*.

Semantically, we say that an expression $R_G\phi$ is true iff ϕ is true in what van Benthem [11, p. 249] calls (G ’s) *communication core*; the model update obtained by removing links to states for members of G that are not linked by *all* members of G . See Fig. 1 for an illustration.

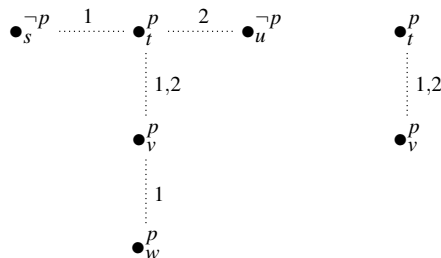


Figure 1: Example taken from [11, p. 248]. Model on the left, its communication core (for the set of all agents $\{1, 2\}$) on the right. Reflexivity, symmetry and transitivity are implicitly assumed.

In this paper we capture that model transformation by the new resolution operators, and study resulting logics. For example, the formula $R_{\{1,2\}}(p \wedge \neg K_1 p)$ will be inconsistent in the resulting logics. $R_{\{1,2\}}(p \wedge K_1 p)$ is true in state t in the model in Fig. 1.

This model transformation abstracts away from the issue of *how* the agents share their information; whether they *communicate* directly with each other and if so in which language, whether they are informed by some outsider about the information other agents have and if so how, and so on. As noted by van Benthem [11, p. 249], the communication core cannot always be obtained by *public announcements* using the epistemic language. Similarly, as noted by several researchers [12, 7, 10], standard distributed knowledge does not always follow logically from the knowledge of the individual agents expressible in the epistemic language. Our model, like that of standard distributed knowledge, is purely semantic: we assume that if an agent can discern between two different worlds, then there exists some mechanism that results in other members of the group being able to make the same distinction. This is further discussed Section 5.

This model transformation models a particular kind of internal group information sharing event. Exactly *which* kind depends on what we assume about what *other* agents, i.e., agents that are not in the group G that resolve their knowledge, know about the fact that this event is taking place. In this paper we will assume that it is common knowledge among the other agents that G resolve their knowledge – but not what the agents in G actually learn. This corresponds to a natural class of events: publicly observable private resolution of distributed knowledge. An example is a meeting in a closed room, where it is observed that a certain group meets in the room to share information.

We want to make it clear that we do not consider distributed knowledge with standard semantics to be “wrong”; the important thing is to be clear about its meaning. In particular, the resolution operators are not intended as a “replacement” of distributed knowledge operators, but as a complement: they express different things. The logics we study contains both types of operators, as well as common knowledge. The main results are sound and complete axiomatizations.

Technically, the model transformation, which amounts to removing certain edges, is similar to those found in the simplest dynamic epistemic logics [5] such as public announcement logics [9]. [11] has also pointed out the close connection between the communication core and sequences of public announcements. Public announcement logics with distributed knowledge have been studied recently [13]. In the absence of *common* knowledge, we get reduction axioms for public announcement logic with distributed knowledge. This turns out to be the case for resolution operators as well. It is not the case in the presence of common knowledge, however.

There is a close connection between the communication core and common knowledge [11]. By studying complete axiomatizations of logics with the resolution operators we make some aspects of that connection precise and give an answer to the question “when does distributed knowledge become common knowledge?” – under certain assumptions.

The rest of the paper is organized as follows. In the next section we review some background definitions and results from the literature, before we introduce logics with the new resolution operators in Section 3, where we also look at some properties of the operators. In Section 4 we prove completeness of resulting logics; the most interesting case being epistemic logic with common and distributed knowledge and resolution operators. We discuss related and future work and conclude in Section 5.

2 Background

In this section we give a (necessarily brief) review of the main background concepts from the literature.

We henceforth assume a countable set of propositional variables PROP and a finite set of agents AG . We let GR be the set of all non-empty groups, i.e., $\text{GR} = \wp(\text{AG}) \setminus \emptyset$.

An *epistemic model* over PROP and AG (or just a *model*) $\mathfrak{M} = (S, \sim, V)$ where S is a set of states (or worlds), $V : \text{PROP} \rightarrow 2^S$ associates a set of states $V(p)$ with each propositional variable p , and \sim is a function that maps each agent to a binary equivalence relation on S . We write \sim_i for $\sim(i)$.

$s \sim_i t$ means that agent i cannot discern between states s and t – if we are in s she doesn’t know whether we are in t , and vice versa. Considering the distributed knowledge of a group G – a key concept in the following – we define a derived relation $\sim_G = \bigcap_{a \in G} \sim_a$ (it is easy to see that \sim_G is an equivalence relation). Intuitively, someone who has all the knowledge of all the members of G can discern between two states if and only if at least one member of G can discern between them. We will also consider common knowledge. A similar relation modeling the common knowledge of a group is obtained by taking the transitive closure of the union of the individual relations: $\sim_{cG} = (\bigcup_{i \in G} \sim_i)^*$.

Definition 1 Below are several languages from the literature.

$$\begin{aligned}
(\mathcal{ELD}) \quad \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid D_G\varphi \\
(\mathcal{ELCD}) \quad \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid D_G\varphi \mid C_G\varphi \\
(\mathcal{PACD}) \quad \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid D_G\varphi \mid C_G\varphi \mid [\varphi]\varphi,
\end{aligned}$$

where $p \in \text{PROP}$, $i \in \text{AG}$ and $G \in \text{GR}$. We use the usual propositional derived operators, as well as $E_G\varphi$ for $\bigwedge_{i \in G} K_i\varphi$.

\mathcal{ELD} and \mathcal{ELCD} are static epistemic languages with distributed knowledge, and with distributed and common knowledge, respectively. These are the languages we will extend with resolution operators in the next section. We will also be interested in \mathcal{PACD} , the language for public announcement logic with both common knowledge and distributed knowledge, when we look at completeness proofs.

Satisfaction of a formula φ of any of these languages in a state m of a model \mathfrak{M} , denoted $\mathfrak{M}, m \models \varphi$, is defined recursively by the following clauses:

$$\begin{aligned}
\mathfrak{M}, m \models p & \quad \text{iff } m \in V(p) \\
\mathfrak{M}, m \models \neg\varphi & \quad \text{iff } \mathfrak{M}, m \not\models \varphi \\
\mathfrak{M}, m \models \varphi \wedge \psi & \quad \text{iff } \mathfrak{M}, m \models \varphi \ \& \ \mathfrak{M}, m \models \psi \\
\mathfrak{M}, m \models K_a\varphi & \quad \text{iff } \forall n \in S. (m \sim_a n \Rightarrow \mathfrak{M}, n \models \varphi) \\
\mathfrak{M}, m \models D_G\varphi & \quad \text{iff } \forall n \in S. (m \sim_G n \Rightarrow \mathfrak{M}, n \models \varphi) \\
\mathfrak{M}, m \models C_G\varphi & \quad \text{iff } \forall n \in S. (m(\bigcup_{i \in G} \sim_i)^* n \Rightarrow \mathfrak{M}, n \models \varphi) \\
\mathfrak{M}, m \models [\psi]\varphi & \quad \text{iff } \mathfrak{M}, m \models \psi \Rightarrow \mathfrak{M}|_{\psi}, m \models \varphi.
\end{aligned}$$

where R^* denotes the transitive closure of R and $\mathfrak{M}|_{\psi}$ is the submodel of \mathfrak{M} restricted to $\{m \in M \mid \mathfrak{M}, m \models \psi\}$. *Validity* is defined as usual: $\models \varphi$ means that $\mathfrak{M}, m \models \varphi$ for all \mathfrak{M} and m .

We now define some axiom schemata and rules. The classical ‘‘S5’’ proof system for multi-agent epistemic logic, denoted **(S5)**, consists of the following axioms and rules:

$$\begin{aligned}
(\text{PC}) \quad & \text{instances of tautologies} \\
(\text{K}) \quad & K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi) \\
(\text{T}) \quad & K_i\varphi \rightarrow \varphi \\
(4) \quad & K_i\varphi \rightarrow K_iK_i\varphi \\
(5) \quad & \neg K_i\varphi \rightarrow K_i\neg K_i\varphi \\
(\text{MP}) \quad & \text{from } \varphi \text{ and } \varphi \rightarrow \psi \text{ infer } \psi \\
(\text{N}) \quad & \text{from } \varphi \text{ infer } K_i\varphi.
\end{aligned}$$

Axioms for distributed knowledge, denoted **(DK)**:

$$\begin{aligned}
(\text{K}_D) \quad & D_G(\varphi \rightarrow \psi) \rightarrow (D_G\varphi \rightarrow D_G\psi) \\
(\text{T}_D) \quad & D_G\varphi \rightarrow \varphi \\
(5_D) \quad & \neg D_G\varphi \rightarrow D_G\neg D_G\varphi \\
(\text{D1}) \quad & K_i\varphi \leftrightarrow D_i\varphi \\
(\text{D2}) \quad & D_G\varphi \rightarrow D_H\varphi, \text{ if } G \subseteq H.
\end{aligned}$$

Axioms and rules for common knowledge, denoted **(CK)**:

$$\begin{aligned}
(\text{K}_C) \quad & C_G(\varphi \rightarrow \psi) \rightarrow (C_G\varphi \rightarrow C_G\psi) \\
(\text{T}_C) \quad & C_G\varphi \rightarrow \varphi \\
(\text{C1}) \quad & C_G\varphi \rightarrow E_G C_G\varphi \\
(\text{C2}) \quad & C_G(\varphi \rightarrow E_G\varphi) \rightarrow (\varphi \rightarrow C_G\varphi) \\
(\text{N}_C) \quad & \text{from } \varphi \text{ infer } C_G\varphi.
\end{aligned}$$

The system that consists of **(S5)** and **(DK)** over the language \mathcal{ELD} , denoted **S5D**, is a sound and complete axiomatization of all \mathcal{ELD} validities. The system that consists of **(S5)**, **(DK)** and **(CK)** over the language \mathcal{ELCD} is a sound and complete axiomatization of all \mathcal{ELCD} validities.

3 Resolving Distributed Knowledge

We want to model the event that G resolves their knowledge. An immediate question is: whenever the group G is a proper subset of the set of all agents, what do *the other* agents know about the fact that this event takes place? Here we will model situations where it is common knowledge among the other agents *that* the event takes place, but not *what* the members of the group learn. As discussed in the introduction, this corresponds to a natural class of information sharing events, namely publicly observable private communication, such as a meeting in a closed room that is observed to be taking place. This is captured by a *global* model update: in every state, remove a link to another state for any member of G whenever it is not the case that there is a link to that state for *all* members of G .

Formally, given a model $\mathfrak{M} = (S, \sim, V)$ and a group of agents G , the (*global*) G -resolved update of \mathfrak{M} is the model $\mathfrak{M}|_G$ where $\mathfrak{M}|_G = (S, \sim|_G, V)$ and

$$(\sim|_G)_i = \begin{cases} \bigcap_{j \in G} \sim_j, & i \in G, \\ \sim_i, & \text{otherwise.} \end{cases}$$

We consider the following new languages with resolution operators.

Definition 2 (Languages)

$$\begin{aligned} (\mathcal{RD}) \quad \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid D_G\varphi \mid R_G\varphi \\ (\mathcal{RCD}) \quad \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid D_G\varphi \mid C_G\varphi \mid R_G\varphi, \end{aligned}$$

where $p \in \text{PROP}$, $i \in \text{AG}$ and $G \in \text{GR}$.

The interpretation of these languages in a pointed model is defined as usual, with the following additional clause for the resolution operator:

$$\mathfrak{M}, s \models R_G\varphi \quad \text{iff} \quad \mathfrak{M}|_G, s \models \varphi.$$

A couple of observations. Recall that we write \sim_H for $\bigcap_{i \in H} \sim_i$. Thus,

$$(\sim|_G)_i = \begin{cases} \sim_G, & i \in G, \\ \sim_i, & i \notin G. \end{cases} \quad (\sim|_G)_H = \begin{cases} \sim_H, & G \cap H = \emptyset, \\ \sim_{G \cup H}, & G \cap H \neq \emptyset. \end{cases}$$

Also note that $(\sim|_G)_i = (\sim|_G)_{\{i\}}$.

3.1 Some Validities

Let us start with a trivial validity: resolution has no effect for a singleton coalition.

Proposition 1 *The following is valid, where $i \in \text{AG}$ and $\varphi \in \mathcal{RCD}$: $R_{\{i\}}\varphi \leftrightarrow \varphi$.*

More interesting are the following properties.

Proposition 2 (Reduction Principles) *The following are valid, where $G, H \in \text{GR}$, $p \in \text{PROP}$ and $\varphi \in \mathcal{RCD}$:*

1. $R_G p \leftrightarrow p$
2. $R_G(\varphi \wedge \psi) \leftrightarrow R_G \varphi \wedge R_G \psi$
3. $R_G \neg \varphi \leftrightarrow \neg R_G \varphi$
4. $R_G K_i \varphi \leftrightarrow D_G R_G \varphi$, when $i \in G$
5. $R_G K_i \varphi \leftrightarrow K_i R_G \varphi$, when $i \notin G$
6. $R_G D_H \varphi \leftrightarrow D_{G \cup H} R_G \varphi$, when $G \cap H \neq \emptyset$
7. $R_G D_H \varphi \leftrightarrow D_H R_G \varphi$, when $G \cap H = \emptyset$.

These properties are reduction principles, of the type known from public announcement logic: they allow us to simplify expressions involving resolution operators. If we have such principles for the combination of resolution with all other operators we can eliminate resolution operators altogether. There are two cases missing above: $R_G C_H$ and $R_G R_H$ ². We consider them next.

3.1.1 Common Knowledge

First, after the *grand* coalition have resolved their knowledge, then all the distributed information in the system is common knowledge: there is no longer a distinction between distributed and common knowledge:

Proposition 3 For any $\varphi \in \mathcal{RCD}$: $R_{AG} C_{AG} \varphi \leftrightarrow R_{AG} D_{AG} \varphi$.

PROOF 3. Given a model $\mathfrak{M} = (S, \sim, V)$ and $s \in S$,

$$\begin{aligned}
& \mathfrak{M}, s \models R_{AG} C_{AG} \varphi \\
& \text{iff } \mathfrak{M}|_{AG}, s \models C_{AG} \varphi \\
& \text{iff } \forall t \in S. (s(\sim|_{AG})_{C_{AG}} t \Rightarrow \mathfrak{M}|_{AG}, s \models \varphi) \\
& \text{iff } \forall t \in S. (s \sim_{AG} t \Rightarrow \mathfrak{M}|_{AG}, s \models \varphi) \quad (\dagger) \\
& \text{iff } \mathfrak{M}|_{AG}, s \models D_{AG} \varphi \\
& \text{iff } \mathfrak{M}, s \models R_{AG} D_{AG} \varphi,
\end{aligned}$$

where for (\dagger) we show that $(\sim|_{AG})_{C_{AG}} = \sim_{AG}$. This is easy: by definition we can verify that for all $i \in AG$, $(\sim|_{AG})_i = \sim_{AG}$; hence $(\sim|_{AG})_{C_{AG}} = (\bigcup_{i \in AG} (\sim|_{AG})_i)^* = (\sim_{AG})^* = \sim_{AG}$.

For the general case, as in the case of distributed knowledge, we have that the resolution operators and common knowledge operators commute when the groups are *disjoint*:

Proposition 4 Let i be an agent, G and H groups of agents and $\varphi \in \mathcal{RCD}$. The following hold:

1. If $G \cap H = \emptyset$, then $\models R_G C_H \varphi \leftrightarrow C_H R_G \varphi$
2. If $G \supseteq H$ and $i \in G$, then $\models R_G C_H \varphi \leftrightarrow R_G K_i \varphi \leftrightarrow D_G R_G \varphi$.

PROOF See the appendix.

However, this does not hold for overlapping groups G and H . In general, we have that (see the proof of the proposition above) $\mathfrak{M}, s \models R_G C_H \varphi$ iff $\mathfrak{M}|_G, t \models \varphi$ for any $(s, t) \in \sim_{H'}^*$, where $\sim_{H'}^* = (\bigcap_{i \in G} \sim_i \cup \bigcup_{i \in H \setminus G} \sim_i)^*$. This does not seem to be reducible.

²The lack of a reduction axiom for the general $R_G R_H \varphi$ case does not mean we cannot get a reduction in the language \mathcal{RCD} : we can simply do the reduction “inside-out”.

3.1.2 Iterated resolution

What about $R_G R_H \varphi$? In extreme cases, we have:

Proposition 5 *The following are valid, where $G, H \in \text{GR}$ and $\varphi \in \mathcal{RCD}$:*

1. $R_G R_H \varphi \leftrightarrow R_H R_G \varphi$, if $G \cap H = \emptyset$
2. $R_G R_G \varphi \leftrightarrow R_G \varphi$.

However, in the general case there does not seem to be a reduction axiom in this case. In particular, $R_G R_H \varphi$ is not equivalent to $R_{G \cup H} \varphi$.

Let us consider an example of iterated resolution.

Example 1 (Triple update) Let $\mathfrak{M} = (S, \sim, V)$ and

$\mathfrak{M}|_{G_1|G_2|G_3} = (S, \sim|_{G_1|G_2|G_3}, V)$. For any agent i , for any number x , we write G_x for “ $i \in G_x$ ”, and $\overline{G_x}$ for “ $i \notin G_x$ ”. Then

$$(\sim|_{G_1|G_2|G_3})_i = \begin{cases} \text{if } \overline{G_1} \overline{G_2} \overline{G_3} : & \sim_i \\ \text{if } G_1 \overline{G_2} \overline{G_3} : & \sim_{G_1} \\ \text{if } \overline{G_1} G_2 \overline{G_3} : & \begin{cases} G_1 \cap G_2 = \emptyset : \sim_{G_2} \\ G_1 \cap G_2 \neq \emptyset : \sim_{G_1 \cup G_2} \end{cases} \\ \text{if } G_1 G_2 \overline{G_3} : & \sim_{G_1 \cup G_2} \\ \text{if } \overline{G_1} G_2 G_3 : & \sim_{G_3} \\ \text{if } G_1 \overline{G_2} G_3 : & \sim_{G_1 \cup G_3} \\ \text{if } \overline{G_1} G_2 G_3 : & \begin{cases} G_1 \cap G_2 = \emptyset : \sim_{G_2 \cup G_3} \\ G_1 \cap G_2 \neq \emptyset : \sim_{G_1 \cup G_2 \cup G_3} \end{cases} \\ \text{if } G_1 G_2 G_3 : & \sim_{G_1 \cup G_2 \cup G_3} \end{cases}$$

In general we get the following (the proof is straightforward from the semantic definition).

Proposition 6 *Let $M = (S, \sim, V)$ and $M|_{G_1|\dots|G_n} = (S, \sim|_{G_1|\dots|G_n}, V)$. Then, following the notation of Example 1, for any $i \in \text{AG}$,*

$$(\sim|_{G_1|\dots|G_n})_i = \begin{cases} \sim_i, & \text{if } \overline{G_1} \dots \overline{G_n} \\ \sim_{G_1 \cup \Theta}, & \text{if starting with } \overline{G_1} G_2 \text{ and} \\ & G_1 \cap G_2 \neq \emptyset \\ \sim_{\Theta}, & \text{otherwise} \end{cases}$$

where Θ is the union of all G_x such that $i \in G_x$.

3.2 Reduction Normal Form for Individual and Distributed Knowledge

As we see from the previous section, the reduction axioms for individual knowledge and distributed knowledge both contain two distinct cases, and the principles of iterated resolution become more complicated. In this section we give a unique form for such reductions, which will be of use later when we prove completeness. We shall call it *reduction normal form* for individual and distributed knowledge.

Definition 3 (δ function) Given an agent i , a group H , and a sequence of groups G_1, \dots, G_n , we define a function δ as follows:

$$\begin{aligned} \delta_0 &= \begin{cases} G_n \cup H, & G_n \cap H \neq \emptyset \\ H, & G_n \cap H = \emptyset \end{cases} \\ \delta_x &= \begin{cases} G_{n-x} \cup \delta_{x-1}, & G_{n-x} \cap \delta_{x-1} \neq \emptyset \\ \delta_{x-1}, & G_{n-x} \cap \delta_{x-1} = \emptyset \end{cases} \\ \delta(H, G_1, \dots, G_n) &= \delta_n. \end{aligned}$$

Clearly $\delta(H, G_1, \dots, G_n) \subseteq H \cup G_1 \cup \dots \cup G_n$. We simply write δ instead of $\delta(H, G_1, \dots, G_n)$ when its parameters are clear in the context.

Proposition 7 *Let $i \in \text{AG}$, $G_1, \dots, G_n, H \in \text{GR}$, $M = (S, \sim, V)$ and $M|_{G_1} | \dots |_{G_n} = (S, \sim|_{G_1} | \dots |_{G_n}, V)$. Then,*

1. $\models R_{G_1} \dots R_{G_n} K_i \varphi \leftrightarrow D_{\delta(\{i\}, G_1, \dots, G_n)} R_{G_1} \dots R_{G_n} \varphi$;
2. $\models R_{G_1} \dots R_{G_n} D_H \varphi \leftrightarrow D_{\delta(H, G_1, \dots, G_n)} R_{G_1} \dots R_{G_n} \varphi$;
3. $(\sim|_{G_1} | \dots |_{G_n})_i = \sim_{\delta(\{i\}, G_1, \dots, G_n)}$;
4. $(\sim|_{G_1} | \dots |_{G_n})_H = \sim_{\delta(H, G_1, \dots, G_n)}$.

PROOF Straightforward: the recursive steps in the definition of the δ function matches exactly the reduction axioms. Note that clauses 1 and 3 can be treated as special cases of clauses 2 and 4 respectively.

4 Axiomatizations

We construct sound and complete axiomatizations of the logics for the two languages \mathcal{RD} and \mathcal{RCD} .

4.1 Resolution and Distributed Knowledge

Consider the language \mathcal{RD} . Let **RD** be the system defined in Figure 2, where **(S5)** and **(DK)** are found in Section 2 and **(RR)** stands for the following reduction axioms for resolution:

- (RA) $R_G p \leftrightarrow p$
- (RC) $R_G(\varphi \wedge \psi) \leftrightarrow R_G \varphi \wedge R_G \psi$
- (RN) $R_G \neg \varphi \leftrightarrow \neg R_G \varphi$
- (RD1) $R_G D_H \varphi \leftrightarrow D_{G \cup H} R_G \varphi$, if $G \cap H \neq \emptyset$
- (RD2) $R_G D_H \varphi \leftrightarrow D_H R_G \varphi$, if $G \cap H = \emptyset$.

Note that **(RR)** contains most of the validities in Proposition 2, except for the reduction principles for individual knowledge – they are provable with RD1, RD2 and D1. In addition, we need the rule N_R for making a reduction to **S5D**. With the rule N_R we can easily show that the rule of *Replacement of Equivalents (RoE)* is admissible in **RD**. RoE allows us to carry out a reduction even without having a reduction axiom for iterated resolution.

<p>(S5) classical proof system for multi-agent epistemic logic</p> <p>(DK) characterization axioms for distributed knowledge</p> <p>(RR) reduction axioms for resolution</p> <p>(N_R) from φ infer $R_G \varphi$</p>
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Figure 2: Axiomatization **RD**.

Theorem 1 *Any \mathcal{RD} formula is valid if and only if it is provable in **RD**.*

4.2 Resolution, Distributed and Common Knowledge

Consider the language \mathcal{RCD} . Let **RCD** be the system defined in Figure 3, which extends **RD** with **(CK)**, found in Section 2, and an induction rule for resolved common knowledge (**RR_C**).

(S5)	classical proof system for multi-agent epistemic logic
(CK)	axioms and rules for common knowledge
(DK)	characterization axioms for distributed knowledge
(N _R)	from φ infer $R_G \varphi$
(RR)	reduction axioms for resolution
(RR _C)	from $\varphi \rightarrow (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$ infer $\varphi \rightarrow R_{G_1} \cdots R_{G_n} C_H \psi$

Figure 3: Axiomatization **RCD**

4.2.1 Soundness

For soundness it suffices to show that the rule RR_C preserves validity (we know that the other axioms/rules are valid/validity preserving from soundness results for the logics based on the sublanguages of \mathcal{RCD}).

Lemma 1 (RR_C-validity preservation) *For all \mathcal{RCD} formulas φ and ψ , all $G_1, \dots, G_n, H \in \text{GR}$, if $\models \varphi \rightarrow (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$, then $\models \varphi \rightarrow R_{G_1} \cdots R_{G_n} C_H \psi$.*

PROOF Suppose $\models \varphi \rightarrow (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$. Given a model \mathfrak{M} and a state s , suppose $\mathfrak{M}, s \models \varphi$, we must show that $\mathfrak{M}, s \models R_{G_1} \cdots R_{G_n} C_H \psi$, i.e., $\mathfrak{M}|_{G_1} \cdots |_{G_n}, s \models C_H \psi$. Thus, for all H -paths $s_0 (\sim |_{G_1} \cdots |_{G_n})_{i_0} \cdots (\sim |_{G_1} \cdots |_{G_n})_{i_{x-1}} s_x$, where $s = s_0$, we need to show that $\mathfrak{M}|_{G_1} \cdots |_{G_n}, s_x \models \psi$.

From $\models \varphi \rightarrow (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$ and $\mathfrak{M}, s_0 \models \varphi$ we get $\mathfrak{M}, s_0 \models (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$, which entails:

$$\mathfrak{M}, s_1 \models \varphi \quad \text{and} \quad \mathfrak{M}|_{G_1} \cdots |_{G_n}, s_0 \models \psi.$$

From $\mathfrak{M}, s_1 \models \varphi$ we get $\mathfrak{M}, s_1 \models (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$, which entails:

$$\mathfrak{M}, s_2 \models \varphi \quad \text{and} \quad \mathfrak{M}|_{G_1} \cdots |_{G_n}, s_1 \models \psi.$$

By similar reasoning, for all $y = 0, \dots, x$, we have

$$\mathfrak{M}, s_y \models \varphi \quad \text{and} \quad \mathfrak{M}|_{G_1} \cdots |_{G_n}, s_y \models \psi,$$

which entails $\mathfrak{M}|_{G_1} \cdots |_{G_n}, s_x \models \psi$ as we wish to show.

Corollary 1 (Soundness) *For any \mathcal{RCD} formula φ , if φ is provable in **RCD**, then it is valid.*

4.2.2 Completeness

As already discussed, **RCD** is similar to **PACD** (axiomatization for public announcement logic with common and distributed knowledge; see [13]): both logics extend epistemic logic with common and distributed knowledge with dynamic operators with update semantics that remove states. There does not seem, however, to be a trivial relationship between the two types of dynamic operators. We are nevertheless able to make heavy use of the completeness proof of **PACD** in [13] when proving completeness of **RCD**. That proof is again based on the completeness proof for public announcement logic with (only) common knowledge found in [2, 5], extended to deal with the distributed knowledge operators (which is non-trivial since intersection is not modally definable). In the following completeness proof we tweak the **PACD** proof to deal with resolution operators instead of public announcement operators. The general proof strategy is as follows: define a finite canonical pseudo model, where distributed knowledge

operators are taken as primitive, and then transform it to a proper model while preserving truth. For the last step we can use a transformation based on *unraveling and folding* in [13] directly.

The most important difference to the **PACD** completeness proof in [13], and indeed the crux of the proof, is the use of the induction rule for resolved common knowledge (RR_C). No corresponding rule is needed in the **PACD** completeness proof. The rule is used in the proof of Lemma 3(8).

Pseudo Semantics

Definition 4 (Pre-models[13]) A *pre-model* is a tuple $\mathfrak{M} = (S, \sim, V)$ where:

- S is a non-empty set of states;
- \sim is a function which maps every agent and every non-empty group of agents to an equivalence relation; we write \sim_i and \sim_G for $\sim(i)$ and $\sim(G)$ respectively;
- $V : \text{PROP} \rightarrow \wp(S)$ is a valuation.

\sim_{C_G} is defined as the reflexive transitive closure of $\bigcup_{i \in G} \sim_i$, just as for a model.

A pre-model is technically a model with a bigger set of agents (all groups are treated as agents in a pre-model). More precisely, if we make the set of agents A explicit in a pre-model, e.g., $\mathfrak{M} = (A, S, \sim, V)$, then \mathfrak{M} is in fact a “genuine” model (S, \sim, V) where the set of agents is $A \cup (\wp(A) \setminus \emptyset)$.

Definition 5 (Pseudo models[13]) A *pseudo model* is a pre-model $\mathfrak{M} = (S, \sim, V)$ such that for any agent i and any groups G and H ,

- $\sim_{\{i\}} = \sim_i$, and
- $G \subseteq H$ implies $\sim_H \subseteq \sim_G$.

A *pointed pre-model* (resp. *pointed pseudo model*) is a tuple (\mathfrak{M}, s) consisted of a pre-model (resp. pseudo model) \mathfrak{M} and a state s in \mathfrak{M} .

Definition 6 (Pseudo semantics) Given a pre-model $\mathfrak{M} = (S, \sim, V)$, let m be a state in M . *Satisfaction at (\mathfrak{M}, s)* is defined as follows:

$$\begin{array}{ll}
\mathfrak{M}, s \models_p p & \text{iff } s \in V(p) \\
\mathfrak{M}, s \models_p \neg \varphi & \text{iff } \mathfrak{M}, s \not\models_p \varphi \\
\mathfrak{M}, s \models_p \varphi \wedge \psi & \text{iff } \mathfrak{M}, s \models_p \varphi \ \& \ \mathfrak{M}, s \models_p \psi \\
\mathfrak{M}, s \models_p K_i \varphi & \text{iff } (\forall t \in \mathfrak{M})(s \sim_i t \Rightarrow \mathfrak{M}, t \models_p \varphi) \\
\mathfrak{M}, s \models_p C_G \varphi & \text{iff } (\forall t \in \mathfrak{M})(s \sim_{C_G} t \Rightarrow \mathfrak{M}, t \models_p \varphi) \\
\mathfrak{M}, s \models_p D_G \varphi & \text{iff } (\forall t \in \mathfrak{M})(s \sim_G t \Rightarrow \mathfrak{M}, t \models_p \varphi) \\
\mathfrak{M}, s \models_p R_G \psi & \text{iff } \mathfrak{M}|_G, s \models_p \psi,
\end{array}$$

where $\mathfrak{M}|_G = (S, \sim|_G, V)$ such that

$$(\sim|_G)_i = \begin{cases} \sim_G, & i \in G \\ \sim_i, & i \notin G \end{cases} \quad \text{and} \quad (\sim|_G)_H = \begin{cases} \sim_{H \cup G}, & H \cap G \neq \emptyset \\ \sim_H, & H \cap G = \emptyset \end{cases}$$

Satisfaction in a pre-model \mathfrak{M} (denoted by $\mathfrak{M} \models_p \varphi$) is defined as usual. We use $\models_p \varphi$ to denote validity, i.e. $\mathfrak{M}, s \models_p \varphi$ for any pointed pre-model (\mathfrak{M}, s) . We write \models instead of \models_p when there is no confusion.

Proposition 8 Let \mathfrak{M} be a pseudo model, G a group of agents. Then $\mathfrak{M}|_G$ is a pseudo model.

PROOF See the appendix.

Proposition 9 *Propositions 6 and 7 still hold for pseudo models.*

When we regard a pre-model as a genuine model, classical (individual) bisimulation becomes an invariance relation. To make this clear, we first elaborate the definition of bisimulation for pre-models, and then introduce its invariance results.

Definition 7 (Pre-model bisimulation) Let two pre-models $\mathfrak{M} = (S, \smile, V)$ and $\mathfrak{M}' = (S', \smile', V')$ be given. A non-empty relation $Z \subseteq S \times S'$ is called a *bisimulation* between \mathfrak{M} and \mathfrak{M}' , denoted by $\mathfrak{M} \rightleftharpoons \mathfrak{M}'$, if for all $\tau \in \text{AG} \cup \text{GR}$, all $s \in S$ and $s' \in S'$ such that sZs' , the following hold.

- (at) For all $p \in \text{PROP}$, $s \in V(p)$ iff $s' \in V'(p)$;
- (zig) For all $t \in S$, if $s \sim_\tau t$, there is a $t' \in S'$ such that $s' \sim'_\tau t'$ and tZt' ;
- (zag) For all $t' \in S'$, if $s' \sim'_\tau t'$, there is a $t \in S$ such that $s \sim_\tau t$ and tZt' .

We say that pointed pre-models (\mathfrak{M}, s) and (\mathfrak{M}', s') are *bisimilar*, denoted $(\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$, if there is a bisimulation Z between \mathfrak{M} and \mathfrak{M}' linking s and s' .

Proposition 10 *Resolution preserves pre-model bisimulation. I.e., for all pointed pre-models (\mathfrak{M}, s) and (\mathfrak{M}', s') , if $(\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$ then $(\mathfrak{M}|_G, s) \rightleftharpoons (\mathfrak{M}'|_G, s')$.*

PROOF See the appendix.

Corollary 2 *For any pre-models \mathfrak{M} and \mathfrak{M}' , if $(\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$ then $\mathfrak{M}, s \models_{\text{p}} \varphi$ iff $\mathfrak{M}', s' \models_{\text{p}} \varphi$ for any \mathcal{RCD} formula φ .*

As introduced in [13], we can also consider a kind of bisimulations between genuine models and pre-models.

Definition 8 (Trans-bisimulation [13]) Let a model $\mathfrak{M} = (M, \sim, V)$ and a pre-model $\mathfrak{N} = (N, \smile, v)$ be given. A non-empty binary relation $Z \subseteq M \times N$ is called a *trans-bisimulation between \mathfrak{M} and \mathfrak{N}* , if for all $m \in M$ and $n \in N$ with mZn :

- (at) $m \in V(p)$ iff $n \in v(p)$ for all $p \in \text{PROP}$,
- (zig_{ag}) For all $m' \in M$ and all $i \in \text{AG}$, if $m \sim_i m'$ (and so $m \sim_{\{i\}} m'$), then there is an $n' \in N$ such that $m'Zn'$ and $n \smile_{\tau_0} \cdots \smile_{\tau_x} n'$ with each of τ_0, \dots, τ_x being “i” or “G” such that $i \in G$;
- (zig_{gr}) For all $m' \in M$ and all $G \in \text{GR}$ with $|G| \geq 2$, if $m \sim_G m'$, then there is an $n' \in N$ such that $m'Zn'$ and $n \smile_{G_1} \cdots \smile_{G_x} n'$ with $G \subseteq G_1 \cap \cdots \cap G_x$;
- (zag) For all $n' \in N$ and all $\tau \in \text{AG} \cup \text{GR}$, if $n \smile_\tau n'$, then there is an $m' \in M$ such that $m'Zn'$ and $m \sim_\tau m'$.

We write $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$ if Z is a trans-bisimulation between \mathfrak{M} and \mathfrak{N} linking m and n . We say a pointed model (\mathfrak{M}, m) and a pointed pre-model (\mathfrak{N}, n) are *trans-bisimilar*, denoted by $(\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$, if there is a trans-bisimulation Z such that $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$.

To make the notation symmetric, we call Z a *trans-bisimulation between \mathfrak{N} and \mathfrak{M}* if it is a trans-bisimulation between \mathfrak{M} and \mathfrak{N} , and we regard $Z : (\mathfrak{N}, n) \rightleftharpoons^T (\mathfrak{M}, m)$ just as $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$.

(Pseudo) satisfaction of \mathcal{RCD} formulas is invariant under trans-bisimulation. We will not prove that directly at this point: it follows from a stronger result we prove later (Lemma 6).

Theorem 2 (Pseudo soundness) *All theorems of \mathcal{RCD} are valid in the class of all pseudo models.*

PROOF See the appendix.

Finitary Canonical Models

Definition 9 (Closure) Given a formula φ , the *closure* of φ is given by the function $cl : \mathcal{RCD} \rightarrow \wp(\mathcal{RCD})$ which is defined as follows:

1. $\varphi \in cl(\varphi)$, and if $\psi \in cl(\varphi)$, so are all of its subformulas;
2. If φ is not a negation, then $\varphi \in cl(\varphi)$ implies $\neg\varphi \in cl(\varphi)$;
3. $K_i\psi \in cl(\varphi)$ iff $D_{\{i\}}\psi \in cl(\varphi)$;
4. $C_G\psi \in cl(\varphi)$ implies $\{K_i C_G\psi \mid a \in A\} \subseteq cl(\varphi)$;
5. $R_{G_1} \cdots R_{G_n} \neg\psi \in cl(\varphi)$ implies $R_{G_1} \cdots R_{G_n} \psi \in cl(\varphi)$;
6. $R_{G_1} \cdots R_{G_n} (\psi \wedge \chi) \in cl(\varphi)$ implies $\{R_{G_1} \cdots R_{G_n} \psi, R_{G_1} \cdots R_{G_n} \chi\} \subseteq cl(\varphi)$;
7. $R_{G_1} \cdots R_{G_n} K_i \psi \in cl(\varphi)$ implies $D_{\delta(\{i\}, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} \psi \in cl(\varphi)$;
8. $R_{G_1} \cdots R_{G_n} D_H \psi \in cl(\varphi)$ implies $D_{\delta(H, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} \psi \in cl(\varphi)$;
9. $R_{G_1} \cdots R_{G_n} C_H \psi \in cl(\varphi)$ implies all of the following:
 - $D_{\delta(H, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} C_H \psi \in cl(\varphi)$,
 - $\{D_{\delta(\{i\}, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} C_H \psi \mid i \in H\} \subseteq cl(\varphi)$,
 - $R_{G_1} \cdots R_{G_n} \psi \in cl(\varphi)$.

It is not hard to verify that the closure of a formula is finite.

We use $\underline{\Gamma}$ as shorthand for $\bigwedge_{\varphi \in \Gamma} \varphi$ when Γ is a finite set of formulas.

Definition 10 (Canonical pseudo model) Let α be a formula. The *canonical pseudo model* $\mathfrak{M}^c = (S, \sim, V)$ for $cl(\alpha)$ is defined below:

- $S = \{\Gamma \mid \Gamma \text{ is maximal consistent in } cl(\alpha)\}$;
- $\Gamma \sim_i \Delta$ iff $\{K_i \varphi \mid K_i \varphi \in \Gamma\} = \{K_i \varphi \mid K_i \varphi \in \Delta\}$;
- $\Gamma \sim_G \Delta$ iff $\{D_H \varphi \mid D_H \varphi \in \Gamma\} = \{D_H \varphi \mid D_H \varphi \in \Delta\}$ whenever $H \subseteq G$;
- $V(p) = \{\Gamma \in S \mid p \in \Gamma\}$.

Proposition 11 *The canonical pseudo model for any $cl(\alpha)$ is a pseudo model.*

PROOF See the appendix.

Lemma 2 *Let $S = \{\Gamma \mid \Gamma \text{ is maximal consistent in } cl(\alpha)\}$ with α a formula. It holds that $\vdash \bigvee_{\Gamma \in S} \underline{\Gamma}$ and $\vdash \varphi \leftrightarrow \bigvee_{\varphi \in \Gamma \in S} \underline{\Gamma}$ for all $\varphi \in cl(\alpha)$.*

PROOF See [5, Exercise 7.16] for the first result (although $cl(\alpha)$ is different in our case the proof is exactly the same). We give a proof of the second result in the appendix.

Let $(S, \sim_{|G_1| \cdots |G_n}, V)$ be an update of a canonical pseudo model, and $\mathfrak{P} = \langle \Phi_0 \succ_{\tau_0} \cdots \succ_{\tau_{n-1}} \Phi_n \rangle$ where \succ stands for $\sim_{|G_1| \cdots |G_n}$ and every τ_x is an agent or a group. If all agents in $\tau_0, \dots, \tau_{n-1}$ appears in H , we call \mathfrak{P} a $\langle G_1 \cdots G_n \rangle$ -resolved H -path (from Φ_0); if a formula φ is such that $\varphi \in \Phi_i$ for all $0 \leq i \leq n$, we call \mathfrak{P} a *canonical φ -path*.

Lemma 3 *If Γ and Δ are maximal consistent in $cl(\alpha)$, then*

1. Γ is deductively closed in $cl(\alpha)$, i.e., $\Gamma \vdash \varphi \Leftrightarrow \varphi \in \Gamma$ for any $\varphi \in cl(\alpha)$;
2. If $\neg\varphi \in cl(\alpha)$, then $\varphi \in \Gamma \Leftrightarrow \neg\varphi \notin \Gamma$;
3. If $\varphi \wedge \psi \in cl(\alpha)$, then $\varphi \wedge \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma \ \& \ \psi \in \Gamma$;
4. If $\underline{\Gamma} \wedge \hat{K}_i \underline{\Delta}$ is consistent, $\Gamma \sim_i \Delta$; if $\underline{\Gamma} \wedge \hat{D}_G \underline{\Delta}$ is consistent, $\Gamma \sim_G \Delta$;
5. If $K_i \varphi \in cl(\alpha)$, then $K_i \Gamma \vdash \varphi \Leftrightarrow K_i \Gamma \vdash K_i \varphi$;

6. If $D_G\varphi \in cl(\alpha)$, then $D_G\Gamma \vdash \varphi \Leftrightarrow D_G\Gamma \vdash D_G\varphi$;
7. If $C_G\varphi \in cl(\alpha)$, then $C_G\varphi \in \Gamma \Leftrightarrow \forall \Delta(\Gamma \sim_{C_G} \Delta \Rightarrow \varphi \in \Delta)$;
8. If $R_{G_1} \cdots R_{G_n} C_H\varphi \in cl(\alpha)$, then $R_{G_1} \cdots R_{G_n} C_H\varphi \in \Gamma$ iff every $\langle G_1 \cdots G_n \rangle$ -resolved H -path from Γ is a canonical $R_{G_1} \cdots R_{G_n} \varphi$ -path.

PROOF We give the proof of the clause 8 in the appendix. Other clauses are the same as in [13, Lemma 49] which can be traced back to [5, Chapter 7].

Lemma 4 (Pseudo truth) Let $\mathfrak{M}^c = (S, \sim, V)$ be the canonical pseudo model for $cl(\alpha)$. For all groups G_1, \dots, G_n , all $\Gamma \in S$, and all $R_{G_1} \cdots R_{G_n} \varphi \in cl(\alpha)$, it holds that

$$R_{G_1} \cdots R_{G_n} \varphi \in \Gamma \quad \text{iff} \quad \mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \varphi.$$

PROOF We show this lemma by induction on φ .

- The base case. $R_{G_1} \cdots R_{G_n} p \in \Gamma$ iff $p \in \Gamma$ (Proposition 2(1)) iff $\mathfrak{M}^c, \Gamma \models p$ iff $\mathfrak{M}^c, \Gamma \models R_{G_1} \cdots R_{G_n} p$ iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models p$.
- The case for negation. $R_{G_1} \cdots R_{G_n} \neg\psi \in \Gamma$ iff $\neg R_{G_1} \cdots R_{G_n} \psi \in \Gamma$ (note that $\neg R_{G_1} \cdots R_{G_n} \psi \in cl(\alpha)$ by Definition 9(2,5)) iff $R_{G_1} \cdots R_{G_n} \psi \notin \Gamma$ iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \not\models \psi$ iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \neg\psi$.
- The case for conjunction. $R_{G_1} \cdots R_{G_n} (\psi \wedge \chi) \in \Gamma$ iff $(R_{G_1} \cdots R_{G_n} \psi \wedge R_{G_1} \cdots R_{G_n} \chi) \in \Gamma$ iff $\{R_{G_1} \cdots R_{G_n} \psi, R_{G_1} \cdots R_{G_n} \chi\} \subseteq \Gamma$ ($R_{G_1} \cdots R_{G_n} \psi$ and $R_{G_1} \cdots R_{G_n} \chi$ are in $cl(\alpha)$) iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \psi$ and $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \chi$ iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \psi \wedge \chi$.
- The case for individual knowledge. From left to right.

$$\begin{aligned} & R_{G_1} \cdots R_{G_n} K_i \psi \in \Gamma \\ \text{iff } & D_\delta R_{G_1} \cdots R_{G_n} \psi \in \Gamma \text{ where } \delta = \delta(\{i\}, G_1, \dots, G_n) \\ \text{iff } & \forall \Delta. (\Gamma \sim_\delta \Delta \Rightarrow D_\delta R_{G_1} \cdots R_{G_n} \psi \in \Delta) \\ \Rightarrow & \forall \Delta. (\Gamma \sim_\delta \Delta \Rightarrow R_{G_1} \cdots R_{G_n} \psi \in \Delta) \quad (\text{T}_D) \\ \text{iff } & \forall \Delta. (\Gamma \sim_\delta \Delta \Rightarrow \mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Delta \models \psi) \quad (\text{IH}) \\ \text{iff } & \forall \Delta. (\Gamma \sim_\delta \Delta \Rightarrow \mathfrak{M}^c, \Delta \models R_{G_1} \cdots R_{G_n} \psi) \\ \text{iff } & \mathfrak{M}^c, \Gamma \models D_\delta R_{G_1} \cdots R_{G_n} \psi \\ \text{iff } & \mathfrak{M}^c, \Gamma \models R_{G_1} \cdots R_{G_n} K_i \psi \quad (7(1), 9) \\ \text{iff } & \mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models K_i \psi. \end{aligned}$$

From right to left. Suppose $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models K_i \psi$. We must show $R_{G_1} \cdots R_{G_n} K_i \psi \in \Gamma$. Suppose this is not the case. Then $\neg R_{G_1} \cdots R_{G_n} K_i \psi \in \Gamma$. Hence $\underline{\Gamma} \wedge \neg R_{G_1} \cdots R_{G_n} K_i \psi$ is consistent, and so is $\underline{\Gamma} \wedge \hat{D}_\delta \neg R_{G_1} \cdots R_{G_n} \psi$, where $\delta = \delta(\{i\}, G_1, \dots, G_n)$. Let \mathcal{S} be the set of all maximal consistent sets in $cl(\alpha)$. By Lemma 2, $\underline{\Gamma} \wedge \hat{D}_\delta \bigvee_{\neg R_{G_1} \cdots R_{G_n} \psi \in \Theta \in \mathcal{S}} \Theta$ is consistent. Since conjunction, resolution and the \hat{D}_δ -operator all distribute over disjunction, $\bigvee_{\neg R_{G_1} \cdots R_{G_n} \psi \in \Theta \in \mathcal{S}} (\underline{\Gamma} \wedge \hat{D}_\delta \Theta)$ is consistent. Therefore there must be a $\Theta \in \mathcal{S}$ such that $\neg R_{G_1} \cdots R_{G_n} \psi \in \Theta$ and $\underline{\Gamma} \wedge \hat{D}_\delta \Theta$ is consistent.

From $\neg R_{G_1} \cdots R_{G_n} \psi \in \Theta$ we get $R_{G_1} \cdots R_{G_n} \psi \notin \Theta$. By the induction hypothesis $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Theta \not\models \psi$, and so $\mathfrak{M}^c, \Theta \not\models R_{G_1} \cdots R_{G_n} \psi$. By Lemma 3(4) and that $\underline{\Gamma} \wedge \hat{D}_\delta \Theta$ is consistent, $\Gamma \sim_\delta \Theta$. But this contradicts the supposition that $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models K_i \psi$, since by the same reasoning as in the proof of the other direction (see above), $\mathfrak{M}^c, \Delta \models R_{G_1} \cdots R_{G_n} \psi$ for all Δ such that $\Gamma \sim_\delta \Delta$.

- The case for distributed knowledge: similar to the case for individual knowledge, and in the proof we use $\delta(H, G_1, \dots, G_n)$ instead of $\delta(\{i\}, G_1, \dots, G_n)$.
- The case for common knowledge. $R_{G_1} \cdots R_{G_n} C_H \psi \in \Gamma$ iff all $\langle G_1 \cdots G_n \rangle$ -resolved H -paths from Γ are also canonical $R_{G_1} \cdots R_{G_n} \psi$ -paths. Namely, for all Δ such that $(\Gamma, \Delta) \in (\sim|_{G_1} | \cdots |_{G_n})_{C_H}$, $R_{G_1} \cdots R_{G_n} \psi \in \Delta$

iff for all Δ such that $(\Gamma, \Delta) \in (\sim|_{G_1} | \cdots |_{G_n})_{C_H}$, $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Delta \models \psi$ (by IH)
 iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models C_H \psi$.

- The case for $R_H \psi$. $R_{G_1} \cdots R_{G_n} R_H \psi \in \Gamma$ iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models \psi$ (IH applies to ψ)
 iff $\mathfrak{M}^c|_{G_1} | \cdots |_{G_n}, \Gamma \models R_H \psi$.

Corollary 3 Let $\mathfrak{M}^c = (S, \sim, V)$ be the canonical pseudo model for $cl(\alpha)$. For all $\Gamma \in S$ and all $\varphi \in cl(\alpha)$, it holds that $\varphi \in \Gamma$ iff $\mathfrak{M}^c, \Gamma \models \varphi$.

Lemma 5 (Pseudo completeness) Let φ be an \mathcal{RCD} -formula. If φ is valid on all pseudo models, then it is provable in \mathcal{RCD} .

From Pseudo Completeness to Completeness By using *unraveling* and *folding* from [13, pp. 9–15], we can transform the canonical pseudo model to a bisimilar pre-model and then to a trans-bisimilar proper model. It remains to show that this process preserves truth. We will use \rightleftharpoons^T to denote the trans-bisimulation relation.

Lemma 6 (Invariance of trans-bisimulation) Let (\mathfrak{M}, m) be a pointed model, (\mathfrak{N}, n) a pointed pre-model, and (\mathfrak{S}, s) a pointed pseudo model. If $(\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n) \rightleftharpoons (\mathfrak{S}, s)$, then $\mathfrak{M}, m \models \varphi$ iff $\mathfrak{N}, n \models_p \varphi$ for all formulas φ .

PROOF The lemma can be shown by induction on φ . Here we only show the case for the resolution operators, proofs of other cases are exactly as in the proof of [13, Lemma 26].

Given a pointed model (\mathfrak{M}, m) , a pointed pre-model (\mathfrak{N}, n) and a pointed pseudo model (\mathfrak{S}, s) , such that $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$ for some Z and $(\mathfrak{N}, n) \rightleftharpoons (\mathfrak{S}, s)$, we have the following:

$$\begin{aligned} \mathfrak{M}, m \models R_G \psi & \text{ iff } \mathfrak{M}|_G, m \models \psi \\ & \text{ iff } \mathfrak{N}|_G, n \models_p \psi \quad (*) \\ & \text{ iff } \mathfrak{N}, n \models_p R_G \psi, \end{aligned}$$

where to show $(*)$ it is sufficient to show that $Z : (\mathfrak{M}|_G, m) \rightleftharpoons^T (\mathfrak{N}|_G, n)$, as $(*)$ is then guaranteed by the induction hypothesis (note that $(\mathfrak{N}|_G, n) \rightleftharpoons (\mathfrak{S}|_G, s)$ by Proposition 10). Let $\mathfrak{M} = (M, \sim, V)$ and $\mathfrak{N} = (N, \sim, \nu)$.

- The case for (at) holds by $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$.
- As for (zig_{gr}), suppose $m(\sim|_G)_H m'$ for some $m' \in M$ and $|H| \geq 2$.
 - If $G \cap H = \emptyset$, $(\sim|_G)_H = \sim_H$. By $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$ there is an $n' \in N$ such that $m' Z n'$ and $n \sim_{H_0} \cdots \sim_{H_x} n'$ with $H \subseteq H_0 \cap \cdots \cap H_x$. Let $H_0 = \cdots = H_x = H$. Thus $n \sim_H \cdots \sim_H n'$. Since $(\sim|_G)_H = \sim_H$, it holds that $n(\sim|_G)_H \cdots (\sim|_G)_H n'$, and so (zig_{gr}) holds in this case.
 - If $G \cap H \neq \emptyset$, $(\sim|_G)_H = \sim_{G \cup H}$. By $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$ there is an $n' \in N$ such that $m' Z n'$ and $n \sim_{H_0} \cdots \sim_{H_x} n'$ with $G \cup H \subseteq H_0 \cap \cdots \cap H_x$. Thus $n \sim_{G \cup H} \cdots \sim_{G \cup H} n'$. Since $(\sim|_G)_H = \sim_{G \cup H}$, It holds that $n(\sim|_G)_H \cdots (\sim|_G)_H n'$. (zig_{gr}) holds also in this case.
- The case for (zig_{ag}) is analogous.
- The case for (zag). For all $n' \in N$ and all $\tau \in \text{AG} \cup \text{GR}$, if $n(\sim|_G)_\tau n'$, then we must show that there is an $m' \in M$ such that $m' Z n'$ and $m(\sim|_G)_\tau m'$.
 - If τ is an agent i . Then if $i \in G$, $(\sim|_G)_i = \sim_G$, otherwise $(\sim|_G)_i = \sim_i$. By $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$, we have $m \sim_G m'$ if $i \in G$, or $m \sim_i m'$ otherwise. Namely $m(\sim|_G)_i m'$ in either case.
 - If τ is a group H . Then if $G \cap H = \emptyset$, $(\sim|_G)_H = \sim_H$, otherwise $(\sim|_G)_H = \sim_{G \cup H}$. By $Z : (\mathfrak{M}, m) \rightleftharpoons^T (\mathfrak{N}, n)$, we have $m \sim_H m'$ if $G \cap H = \emptyset$, or $m \sim_{G \cup H} m'$ otherwise. Namely $m(\sim|_G)_H m'$ in either case.

We have shown that the lemma holds for the case for resolution. For other cases we refer to the proof of [13, Lemma 26].

Theorem 3 (Completeness) *For any RCD formula φ , if φ is valid then it is provable in RCD.*

PROOF It suffices to show that any RCD-consistent formula is satisfiable. Let φ be consistent. Let \mathfrak{M}^c be the canonical pseudo model for $cl(\varphi)$. By the pseudo truth lemma (with $n = 0$, i.e., an empty list of resolution operators), φ is satisfied in a state Γ in \mathfrak{M}^c . Now let $\mathfrak{N}^{\mathfrak{M}^c}$ be the *unraveling* [13, Definition 18]³ of \mathfrak{M}^c . $\mathfrak{N}^{\mathfrak{M}^c}$ is a pre-model [13, Proposition 19]. Now let $(\mathfrak{N}^{\mathfrak{M}^c})^*$ be the *folding* [13, Definition 22] of $\mathfrak{N}^{\mathfrak{M}^c}$. $(\mathfrak{N}^{\mathfrak{M}^c})^*$ is a (proper) model [13, Definition 22]. From [13, Lemma 27] and [13, Lemma 28] we have that unraveling preserves bisimulation and that folding preserves trans-bisimulation, in other words we have that⁴ $(\mathfrak{M}, \Gamma) \rightleftharpoons (\mathfrak{N}^{\mathfrak{M}^c}, \bar{\Gamma}) \rightleftharpoons^T ((\mathfrak{N}^{\mathfrak{M}^c})^*, \bar{\Gamma})$. By Corollary 2, $(\mathfrak{N}^{\mathfrak{M}^c}, \bar{\Gamma}) \models_p \varphi$. By Lemma 6, $((\mathfrak{N}^{\mathfrak{M}^c})^*, \bar{\Gamma}) \models \varphi$ and we are done.

5 Discussion

In this paper we captured the dynamics of publicly observable private resolution of distributed knowledge. Resolution operators (using update semantics) are both an alternative and a complement to the standard distributed knowledge operators (which use standard modal semantics).

Resolution operators let us reason about the relationship between common knowledge and distributed knowledge in general, and in particular about distributed knowledge as potential common knowledge – when can distributed knowledge become common knowledge? A naive idea would be that $D_G\varphi$ should imply that $R_G C_G \varphi$ – any information that is distributed can become common knowledge through resolution. This does not hold in general, however, due to Moore-like phenomena – φ might even become false after resolution (an example is the formula $D_{\{1,2\}}(p \wedge \neg K_1 p)$ discussed in the introduction). We do, however, have the following (Prop. 4(2) with $G = H$):

$$R_G C_G \varphi \leftrightarrow D_G R_G \varphi.$$

A fact can become common knowledge after the group have shared their information if and only if it was distributed knowledge before the event that the fact would be true after the event. This is exactly the distributed knowledge that can become common knowledge (in our special case of publicly observable private resolution of distributed knowledge). If the grand coalition resolves its distributed knowledge, there is no distinction between distributed and common knowledge any more: $R_{AG} C_{AG} \varphi \leftrightarrow R_{AG} D_{AG} \varphi$ (Prop. 3).

As discussed in the introduction, it has been argued that distributed knowledge in general does not comply with the following *principle of full communication* [12]: if $D_G\varphi$ is true, then φ follows logically from the set of all formulas known by at least one agent in the group. This is seen as a problem: namely that agents can have distributed knowledge without being able to establish it “through communication” [12]. Several papers [12, 7, 10] have tried to characterize classes of models on which the principle of full communication *does* hold – the class of all such models is called *full communication models* [10]. This may seem related to the distinction between distributed knowledge and resolution operators: the latter is intuitively related to internal “full” communication in the group. However, this similarity is superficial: the notion of full communication in the sense of [12] is about *expressive power of the*

³While unraveling is a standard general technique; here we mean unraveling exactly in the sense of the mentioned definition.

⁴Here $\bar{\Gamma}$ is any path in the unraveling starting with Γ .

communication language and the limits that puts on the resulting possible epistemic states under certain assumptions about how information is shared. The key point of the resolution operators, on the other hand, compared with the standard distributed knowledge operators, is to make a distinction between *before* and *after* the information sharing event. That distinction is not made in standard distributed knowledge – even restricted to full communication models: it is easy to see that, e.g., $D_{\{1,2\}}(p \wedge \neg K_1 p)$ is satisfiable also on full communication models. The two ideas, of limiting models to full communication models and of modeling group information sharing events using model updates, are orthogonal, and there is nothing against restricting logics with the resolution operators to full communication models. We leave that for future work. Furthermore, it would be interesting to look at a combined variant: “update by full communication”, which takes the communication language into account when defining the updated model.

A main interest for future work is expressive power. Can it be shown that \mathcal{RCD} is strictly more expressive than \mathcal{ELCD} ? Another, related, natural question is the relative expressivity of \mathcal{RCD} and \mathcal{PACD} : can the combination of public announcement operators (which eliminate states) and distributed knowledge operators (which pick out states considered possible by everyone) always be used to “simulate” the resolution operators (which eliminate states considered possible by everyone)?

Also of interest for future work is to look at other assumptions about the other agents’ knowledge about the group communication event taking place. In this paper we only studied the case that it is common knowledge that the event takes place (but not what the agents in the group learn). That was naturally modeled using a “global” model update: in every state, replace accessibility for each agent in the group with the group accessibility (intersection). An interesting and also natural alternative is doing only a “local” model update: change accessibility in the same way, but only in the current state. That would correspond to it being common knowledge that if this is the current state, then the group resolves their knowledge.

When looking at the interaction of the resolution and common knowledge operators one might be reminded of *relativized common knowledge* [3, 4]. Here is an open question: can $R_G C_H \varphi$ be expressed using relativized common knowledge, in combination with other operators?

Finally, there is a conceptual relationship to *group announcement logic* [1], where formulas of the form $\langle G \rangle \varphi$ say that G can make a joint public announcement such that φ will become true. A difference to the resolution operators in this paper is that latter model private communication. Yet, the exact relationship between these operators is interesting for future work.

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A Some proofs

Proof of Proposition 4 1. $\mathfrak{M}, s \models R_G C_H \varphi$ iff $\mathfrak{M}|_G, s \models C_H \varphi$ iff $\mathfrak{M}|_G, t \models \varphi$ for any $(s, t) \in \sim_H^*$, where $\sim_H^* = (\bigcup_{i \in H} \sim_i)^*$ and $\sim_i = \bigcap_{j \in G} \sim_j$ for $i \in G$ and $\sim_i = \sim_i$ for $i \notin G$. Thus, when $G \cap H = \emptyset$, we get that $\sim_H^* = (\bigcup_{i \in H} \sim_i)^*$. $\mathfrak{M}|_G, t \models \varphi$ for any $(s, t) \in (\bigcup_{i \in H} \sim_i)^*$ holds iff $\mathfrak{M}, t \models R_G \varphi$ for any $(s, t) \in (\bigcup_{i \in H} \sim_i)^*$ iff $\mathfrak{M}, t \models C_H R_G \varphi$.

2.

$$\begin{aligned}
& \mathfrak{M}, s \models R_G C_H \varphi \\
\text{iff } & \mathfrak{M}|_G, s \models C_H \varphi \\
\text{iff } & \mathfrak{M}|_G, t \models \varphi \text{ for all } t \text{ s.t. } (s, t) \in \sim_{C_H}^G \\
\text{iff }^\dagger & \mathfrak{M}|_G, t \models \varphi \text{ for all } t \text{ s.t. } (s, t) \in \sim_i^G \\
\text{iff } & \mathfrak{M}|_G, s \models K_i \varphi \\
\text{iff } & \mathfrak{M}, s \models R_G K_i \varphi
\end{aligned}$$

For the \dagger step, note that when $i \in G$, $\sim_i^G = \sim_j^G$ for any $j \in G$ (and actually also equal to \sim_G). Therefore,

$$\sim_{C_H}^G = \left(\bigcup_{i \in H} \sim_i^G \right)^* = (\sim_i^G)^* = \sim_i^G.$$

That $R_G K_i \varphi \leftrightarrow D_G R_G \varphi$ is valid is already shown in Proposition 2.

Proof of Proposition 8 Let $\mathfrak{M} = (S, \sim, V)$. Clearly $\mathfrak{M}|_G = (S, \sim|_G, V)$ is a pre-model. Moreover,

1. Given an agent i ,

$$\begin{aligned} (\sim|_G)_{\{i\}} &= \begin{cases} \sim_{\{i\} \cup G} & i \in G \\ \sim_{\{i\}} & i \notin G \end{cases} \\ &= \begin{cases} \sim_G & i \in G \\ \sim_i & i \notin G \end{cases} \\ &= (\sim|_G)_i. \end{aligned}$$

2. Given two groups H and H' such that $H \subseteq H'$,

$$\begin{aligned} (\sim|_G)_{H'} &= \begin{cases} \sim_{H' \cup G} & H' \cap G \neq \emptyset \\ \sim_{H'} & H' \cap G = \emptyset \end{cases} \\ (\sim|_G)_H &= \begin{cases} \sim_{H \cup G} & H \cap G \neq \emptyset \\ \sim_H & H \cap G = \emptyset \end{cases} \end{aligned}$$

So we have:

- when $H \cap G \neq \emptyset$ (and therefore $H' \cap G \neq \emptyset$), $(\sim|_G)_{H'} = \sim_{H' \cup G} \subseteq \sim_{H \cup G} = (\sim|_G)_H$;
- when $H' \cap G = \emptyset$ (and therefore $H \cap G = \emptyset$), $(\sim|_G)_{H'} = \sim_{H'} \subseteq \sim_H = (\sim|_G)_H$;
- otherwise $H' \cap G \neq \emptyset$ and $H \cap G = \emptyset$, and in this case $(\sim|_G)_{H'} = \sim_{H' \cup G} \subseteq \sim_H = (\sim|_G)_H$.

$\mathfrak{M}|_G$ is a pre-model satisfying the two conditions above, which shows it is a pseudo model.

Proof of Proposition 10 Let $\mathfrak{M} = (S, \sim, V)$ and $\mathfrak{M}' = (S', \sim', V')$. Thus $\mathfrak{M}|_G = (S, \sim|_G, V)$ and $\mathfrak{M}'|_G = (S', \sim'|_G, V')$. Suppose $Z : (\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$, and we show $Z : (\mathfrak{M}|_G, s) \rightleftharpoons (\mathfrak{M}'|_G, s')$:

(at) This clearly follows from the (at) clause of $Z : (\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$.

(zig) For all $t \in S$, if $s(\sim|_G)Ht$, then

- If $G \cap H = \emptyset$, then $(\sim|_G)_H = \sim_H$ and $(\sim'|_G)_H = \sim'_H$. By $Z : (\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$ there must be a $t' \in S'$ such that $s'(\sim'|_G)Ht'$ and tZt' .
- If $G \cap H \neq \emptyset$, then $(\sim|_G)_H = \sim_{G \cup H}$ and $(\sim'|_G)_H = \sim'_{G \cup H}$. By $Z : (\mathfrak{M}, s) \rightleftharpoons (\mathfrak{M}', s')$ there must be a $t' \in S'$ such that $s'(\sim'|_G)Ht'$ and tZt' .

If $s(\sim|_G)it$, we can prove analogously to the above.

(zag) This can be shown analogously to the case for (zig).

Proof of Theorem 2 It is easy to verify that (S5), (CK), (DK), (N_R), (RA), (RC) and (RN) are all valid or admissible with respect to the class of all pseudo models. Here we only show that i) (RD1) and (RD2) are valid in all pseudo models, and ii) (RR_C) preserves validity of pseudo models.

Let $\mathfrak{M} = (S, \sim, V)$ be a pseudo model and $s \in S$. We show the following:

- $\mathfrak{M}, s \models_{\text{p}} \text{RD1}$ and $\mathfrak{M}, s \models_{\text{p}} \text{RD2}$, i.e.,
 - If $G \cap H \neq \emptyset$, then $\mathfrak{M}, s \models_{\text{p}} R_G D_H \varphi \leftrightarrow D_{G \cup H} R_G \varphi$;
 - If $G \cap H = \emptyset$, then $\mathfrak{M}, s \models_{\text{p}} R_G D_H \varphi \leftrightarrow D_H R_G \varphi$.
- $$\begin{aligned} &\mathfrak{M}, s \models_{\text{p}} R_G D_H \varphi \\ \text{iff } &\mathfrak{M}|_G, s \models_{\text{p}} D_H \varphi \\ \text{iff } &\mathfrak{M}|_G, t \models_{\text{p}} \varphi \text{ for all } t \text{ s.t. } (s, t) \in (\sim|_G)_H \\ \text{iff } &\mathfrak{M}, t \models_{\text{p}} R_G \varphi \text{ for all } t \text{ s.t. } (s, t) \in (\sim|_G)_H \\ \text{iff }^\dagger &\text{ if } G \cap H \neq \emptyset, \mathfrak{M}, t \models_{\text{p}} R_G \varphi \text{ for all } t \text{ s.t. } (s, t) \in \sim|_{G \cup H}, \\ &\text{ if } G \cap H = \emptyset, \mathfrak{M}, t \models_{\text{p}} R_G \varphi \text{ for all } t \text{ s.t. } (s, t) \in \sim|_H \\ \text{iff } &\text{ if } G \cap H \neq \emptyset, \mathfrak{M}, t \models_{\text{p}} D_{G \cup H} R_G \varphi, \text{ and} \\ &\text{ if } G \cap H = \emptyset, \mathfrak{M}, t \models_{\text{p}} D_H R_G \varphi, \end{aligned}$$

where the \dagger step is by definition:

$$(\sim|_G)_H = \begin{cases} \sim_{H \cup G}, & G \cap H \neq \emptyset, \\ \sim_H, & G \cap H = \emptyset. \end{cases}$$

- $\mathfrak{M}, s \models_p \varphi \rightarrow R_{G_1} \cdots R_{G_n} C_H \psi$ under the assumption $\models_p \varphi \rightarrow (E_H \varphi \wedge R_{G_1} \cdots R_{G_n} \psi)$. The proof is similar to the proof for genuine models.

Proof of Proposition 11 Suppose that $\mathfrak{M} = (S, \sim, V)$ is the canonical pseudo model for $cl(\alpha)$. We need to show that \mathfrak{M} is a pseudo model. Namely,

1. S is non-empty, and
2. all \sim_i 's and \sim_G 's are equivalence relations, and
3. V is a valuation from PROP to $\wp(S)$, and
4. $\sim_i = \sim_{\{i\}}$ for every agent i , and
5. $\sim_H \subseteq \sim_G$ if G and H are groups such that $G \subseteq H$.

Conditions 1–3 are the conditions for being a pre-model which are easy to verify. Conditions 4 and 5 are additional conditions for being a pseudo model.

By Definition 9(3), $K_i \varphi$ and $D_i \varphi$ must be in $cl(\alpha)$ both or neither. Thus, for any $\Gamma, \Delta \in S$,

$$\begin{aligned} & \Gamma \sim_i \Delta \\ \text{iff } & \{K_i \varphi \mid K_i \varphi \in \Gamma\} = \{K_i \varphi \mid K_i \varphi \in \Delta\} \\ \text{iff } & \{D_i \varphi \mid D_i \varphi \in \Gamma\} = \{D_i \varphi \mid D_i \varphi \in \Delta\} \quad (\text{Axiom DK1}) \\ \text{iff } & \Gamma \sim_{\{i\}} \Delta. \end{aligned}$$

$$\begin{aligned} & \Gamma \sim_H \Delta \\ \text{iff } & \{D_{H'} \varphi \mid D_{H'} \varphi \in \Gamma\} = \{D_{H'} \varphi \mid D_{H'} \varphi \in \Delta\}, \\ & \text{for any group } H' \subseteq H \\ \Rightarrow & \{D_{G'} \varphi \mid D_{G'} \varphi \in \Gamma\} = \{D_{G'} \varphi \mid D_{G'} \varphi \in \Delta\}, \\ & \text{for any group } G' \subseteq G \\ \text{iff } & \Gamma \sim_G \Delta. \end{aligned}$$

This finishes the proof, and shows that the notion “canonical pseudo model” is well-defined.

Proof of Lemma 2(2) Let $\varphi \in cl(\alpha)$. By $\vdash (\bigvee_{\neg\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma}) \rightarrow \neg\varphi$ and the first result of this lemma (i.e., $\vdash \bigvee_{\neg\varphi \in \Gamma \in \mathcal{S}} \bigvee_{\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma}$) we get $\vdash \varphi \rightarrow \bigvee_{\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma}$. For the converse direction, suppose $\not\vdash \bigvee_{\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma} \rightarrow \varphi$. Then $\neg(\bigvee_{\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma} \rightarrow \varphi)$ is consistent. Namely $\neg\varphi \wedge \bigvee_{\varphi \in \Gamma \in \mathcal{S}} \underline{\Gamma}$ is consistent. But this is impossible.

Proof of Lemma 3(8) Let $R_{G_1} \cdots R_{G_n} C_H \varphi \in cl(\alpha)$. It follows from the definition of closure (Definition 9) that the following formulas:

- $D_{\delta(\{i\}, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} C_H \varphi$ where $i \in H$
- $D_{\delta(H, G_1, \dots, G_n)} R_{G_1} \cdots R_{G_n} C_H \varphi$
- $R_{G_1} \cdots R_{G_n} \varphi$ and $\neg R_{G_1} \cdots R_{G_n} \varphi$

are all in $cl(\alpha)$.

From left to right. Suppose $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma$, we continue by induction on the length of the path that every $\langle G_1 \cdots G_n \rangle$ -resolved H -path from Γ is a canonical $R_{G_1} \cdots R_{G_n} C_H \varphi$ -path. Then the left-to-right

direction follows: by $\vdash C_H \varphi \rightarrow \varphi$, N_R and R_G -distribution (which follows from **RR** axioms) we get $\vdash R_{G_1} \cdots R_{G_n} C_H \varphi \rightarrow R_{G_1} \cdots R_{G_n} \varphi$, and by $R_{G_1} \cdots R_{G_n} \varphi \in cl(\alpha)$ we have $R_{G_1} \cdots R_{G_n} \varphi \in \Gamma$.

Suppose the length of the $\langle G_1 \cdots G_n \rangle$ -resolved H -path is 0, i.e., the path is $\langle \Gamma \rangle$, we must show that $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma$. This is guaranteed by the supposition.

Suppose the length of the $\langle G_1 \cdots G_n \rangle$ -resolved H -path is $n + 1$, i.e., the path is $\langle \Gamma_0 \succ_{\tau_0} \cdots \succ_{\tau_{n-1}} \Gamma_n \succ_{\tau_n} \Gamma_{n+1} \rangle$ with $\Gamma_0 = \Gamma$ and every τ_x is either in H or a subset of H . By the induction hypothesis we may assume that $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_n$.

- Suppose τ_n is an agent i ($i \in H$). By Axiom C1 we have $\vdash C_H \varphi \rightarrow K_i C_H \varphi$. It follows that $\vdash R_{G_1} \cdots R_{G_n} C_H \varphi \rightarrow R_{G_1} \cdots R_{G_n} K_i C_H \varphi$ by the rules N_R and R_G -distribution. Let $\delta = \delta(\{i\}, G_1, \dots, G_n)$. By the reduction axioms we move K_i left, i.e., $\vdash R_{G_1} \cdots R_{G_n} K_i C_H \varphi \rightarrow D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi$, so we get $\vdash R_{G_1} \cdots R_{G_n} C_H \varphi \rightarrow D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi$. Hence $\Gamma_n \vdash D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi$. As $D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi \in cl(\alpha)$, we have $D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_n$. Moreover, by Proposition 9, $\succ_i = \sim_\delta$. Thus $D_\delta R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_{n+1}$ by the definition of \sim_δ , and so $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_{n+1}$.
- Suppose τ_n is a group I ($I \subseteq H$). By Axioms C1, D1 and D2 we have $\vdash C_H \varphi \rightarrow D_I C_H \varphi$. By N_R and R_G -distribution, $\vdash R_{G_1} \cdots R_{G_n} C_H \varphi \rightarrow R_{G_1} \cdots R_{G_n} D_I C_H \varphi$. By similar reasoning to the case above, we get the result $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_{n+1}$ (we use $\delta(H, G_1, \dots, G_n)$ instead of $\delta(\{i\}, G_1, \dots, G_n)$ in this case).

In both cases we get $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma_{n+1}$ as we wish to show.

From right to left. Suppose that every $\langle G_1 \cdots G_n \rangle$ -resolved H -path from Γ is a canonical $R_{G_1} \cdots R_{G_n} \varphi$ -path. Let \mathcal{S}_0 be the set of all maximal consistent sets Δ in $cl(\alpha)$ such that every $\langle G_1 \cdots G_n \rangle$ -resolved H -path from Δ is a canonical $R_{G_1} \cdots R_{G_n} \varphi$ -path. Now consider the formula

$$\lambda = \bigvee_{\Delta \in \mathcal{S}_0} \Delta$$

We will show the following:

1. $\vdash \underline{\Gamma} \rightarrow \lambda$
2. $\vdash \lambda \rightarrow (E_H \lambda \wedge R_{G_1} \cdots R_{G_n} \varphi)$.

From the above and the reduction rule for resolved common knowledge we get $\vdash \underline{\Gamma} \rightarrow R_{G_1} \cdots R_{G_n} C_H \varphi$ which furthermore entails $R_{G_1} \cdots R_{G_n} C_H \varphi \in \Gamma$. We now continue with the proof of the two clauses.

1. This is trivial, as $\underline{\Gamma}$ is one of the disjuncts of λ .
2. Suppose towards a contradiction that

$$\lambda \wedge \neg(E_H \lambda \wedge R_{G_1} \cdots R_{G_n} \varphi)$$

is consistent, i.e., $\lambda \wedge (\neg E_H \lambda \vee \neg R_{G_1} \cdots R_{G_n} \varphi)$ is consistent. Because λ is a disjunction there must be a disjunct $\underline{\Xi}$ of λ such that $\underline{\Xi} \wedge (\neg E_H \lambda \vee \neg R_{G_1} \cdots R_{G_n} \varphi)$ is consistent. It follows that either $\underline{\Xi} \wedge \neg E_H \lambda$ or $\underline{\Xi} \wedge \neg R_{G_1} \cdots R_{G_n} \varphi$ is consistent.

If the former is consistent, then there must be an agent $i \in H$ such that $\underline{\Xi} \wedge \neg K_i \lambda$ is consistent, i.e., $\underline{\Xi} \wedge \hat{K}_i \neg \bigvee_{\Delta \in \mathcal{S}_0} \Delta$ is consistent. Since $\vdash \bigvee_{\Delta \in \mathcal{S}} \Delta$ by Lemma 2, we have $\vdash \neg \bigvee_{\Delta \in \mathcal{S}_0} \Delta \rightarrow \bigvee_{\Delta' \in \mathcal{S} \setminus \mathcal{S}_0} \Delta'$, and so there must be a Θ in $\mathcal{S} \setminus \mathcal{S}_0$ such that $\underline{\Xi} \wedge \hat{K}_i \Theta$ is consistent. By item 4 of this lemma $\underline{\Xi} \sim_i \Theta$ (where \sim is the relation in the canonical pseudo model for $cl(\alpha)$). But then $\underline{\Xi}$ cannot be in \mathcal{S}_0 for $\Theta \notin \mathcal{S}_0$. A contradiction!

If the latter is consistent, since $\neg R_{G_1} \cdots R_{G_n} \varphi \in cl(\alpha)$ and $\underline{\Xi}$ is maximal, $\neg R_{G_1} \cdots R_{G_n} \varphi \in \underline{\Xi}$. But $R_{G_1} \cdots R_{G_n} \varphi \in \underline{\Xi}$ since $\underline{\Xi} \sim_H \underline{\Xi}$ and every $\langle G_1 \cdots G_n \rangle$ -resolved H -path from $\underline{\Xi}$ is a canonical $R_{G_1} \cdots R_{G_n} \varphi$ -path. We reach a contradiction.