The Computational Complexity of Choice Sets*

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Abstract

Social choice rules are often evaluated and compared by inquiring whether they fulfill certain desirable criteria such as the *Condorcet criterion*, which states that an alternative should always be chosen when more than half of the voters prefer it over any other alternative. Many of these criteria can be formulated in terms of choice sets that single out reasonable alternatives based on the preferences of the voters. In this paper, we consider choice sets whose definition merely relies on the pairwise majority relation. These sets include the *Copeland set*, the *Smith set*, the *Schwartz set*, *von Neumann-Morgenstern stable sets* (a concept originally introduced in the context of cooperative game theory), the *Banks set*, and the *Slater set*. We investigate the relationships between these sets and completely characterize their computational complexity which allows us to obtain hardness results for entire classes of social choice rules. In contrast to most existing work, we do not impose any restrictions on individual preferences, apart from the indifference relation being reflexive and symmetric. This assumption is motivated by the fact that many realistic types of preferences in computational contexts such as incomplete or quasi-transitive preferences may lead to general pairwise majority relations that need not be complete.

1 INTRODUCTION

Given a profile of individual preferences over a number of alternatives, the simple majority rule—choosing the alternative which the majority of agents prefer over the other alternative—is an attractive way of aggregating social preferences over any pair of alternatives. It has an intuitive appeal to democratic principles, is simple to understand and, most importantly, has some formally attractive properties. May's theorem shows that a number of rather weak and intuitively acceptable principles completely characterize the majority rule in settings with two alternatives (see May, 1952). Moreover, almost all common social choice rules satisfy May's axioms and thus coincide with the majority rule in the two alternative case. Thus, it would seem that the existence of a majority of individuals preferring alternative a to alternative b signifies something fundamental and generic about the group's preferences over a and b. We will say that in any such case alternative *a dominates* alternative b.

Based on the simple majority rule, this dominance relation is obviously *asymmetric* in the strong sense that a dominating b implies that b does not dominate a. *A fortiori* the dominance relation is also irreflexive, *i.e.*, no alternative dominates itself. Conversely, any asymmetric binary relation on the set of alternatives, is induced as the dominance relation of some preference profile, provided that the number of voters is large enough compared to the number of alternatives (McGarvey, 1953). As is well known from Condorcet's paradox (de Condorcet, 1785), however, the dominance relation may very well contain cycles. This implies that the dominance relation need not have a maximum, or even a maximal, element, even if the underlying individual preferences do. Thus, the concept of maximality is rendered untenable in most cases.

There are several ways to get around this problem. One of which is, of course, to abandon the simple majority rule altogether. We will not consider such attempts here. Another would be to take more structure of the underlying individual preference profiles into account. We will not consider these here either. A third way out is to take the dominance relation for granted and define alternative concepts to take over the role of the maximality. As such we will be concerned with criteria for social choice correspondences that are based on the dominance relation only, *i.e.*, those that Fishburn (1977) called $C1$ functions. Formally, by a $C1$

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social choice concept we will understand a concept that is invariant for all preference profiles that give rise to the same dominance relation. Examples of such concepts are *the Condorcet winner*, defined as the alternative, if any, that dominates all other alternatives. Other examples are:

- the *Copeland set*, *i.e.*, the set of all alternatives for which the difference between the number of alternatives it dominates and the number of alternatives that it is dominated by is maximal,
- the *Smith set*, *i.e.*, the smallest set of alternatives that dominate all alternatives that are not in the set,
- the *Schwartz set*, *i.e.*, the union of all minimal sets of alternatives that are not dominated by any alternative outside that set,
- *von Neumann-Morgenstern stable sets*, *i.e.*, any set U consisting precisely of those alternatives that are not dominated by any alternative in U,
- the *Banks set*, *i.e.*, the set of maximal elements of complete and transitive dominance subrelations induced by a maximal subset of alternatives (with respect to subset inclusion), and
- the *Slater set*, *i.e.*, the set of undominated elements of those acyclic relations that share as many tuples with the original dominance relation as possible.

Social choice literature often mentions that one choice rule "is more difficult to compute" than another. The main goal of this paper is to provide formal grounds for such statements and, in particular, to obtain lower bounds for the computational complexity of entire classes of choice functions. This approach is inspired by Bartholdi, III et al. (1989), who proved the NP-hardness of any social *welfare* functional that is neutral, consistent, and Condorcet. They admit that "since only the Kemeny rule satisfies the hypotheses, this corollary is not entirely satisfying" (Bartholdi, III et al., 1989). Over the last few years, the computational complexity of various existing voting rules (such as the Dodgson rule, the Kemeny rule, or the Young rule) has been completely characterized (see Faliszewski et al., 2006, for a recent survey). However, we are not aware of any hardness results regarding broader classes of rules.[1]

It is interesting to note that social choice theory literature almost exclusively deals with *tournaments*, *i.e.*, asymmetric and complete relations on a set of alternatives. For any odd number of *linear* individual preferences, the simple majority dominance relation is indeed a tournament. From a social choice perspective these could be taken as relatively mild and technically convenient restrictions. For one, the transitivity of a tournament implies its acyclicity and *vice versa*. Moreover, there can be at most one maximal element in a tournament, and if there is one it is the *Condorcet winner*. Without these restrictions, the simple majority rule allows for ties and the dominance relation need not be complete. From the perspective of computational complexity, however, the restriction to tournaments is not as harmless as it might seem from a social choice point of view. We will find that some problems we consider are computationally significantly easier for tournaments than for the general case. Furthermore, in settings of computational interest such as webpage ranking there is usually a large number of alternatives over which the voters only have partial preferences with possibly many indifferences (see *e.g.*, Altman and Tennenholtz, 2005).

The remainder of this paper is structured as follows. The social choice setting we consider is introduced in Section 2. Section 3 motivates, introduces, and analyzes six choice sets whose computational complexity is investigated in Section 4. Section 5 concludes the paper with an overview and interpretation of the results.

2 PRELIMINARIES

In social choice theory, agents from a finite set N choose among a finite set A of alternatives (see *e.g.*, Sen, 1969; Fishburn, 1973; Arrow et al., 2002). The cardinalities of these sets will be denoted n and m, respectively. For each agent $i \in N$ there is a binary preference relation $\succsim_i$ over the alternatives in A. We have $a \succsim_i b$ denote that player i values alternative a at least as much as alternative b. As usual, we write $\succ_i$ for the strict part of $\succsim_i$, *i.e.*, $a \succ_i b$ if $a \succsim_i b$ but not $b \succsim_i a$. Similarly, $\sim_i$ denotes i's indifference relation, *i.e.*, $a \sim_i b$ if both $a \succsim_i b$ and $b \succsim_i a$. We make no specific structural assumptions individual preferences should fulfill, apart from the indifference relation being reflexive and symmetric. Obviously, this includes all *linear orders*—*i.e.*, reflexive, transitive, complete and anti-symmetric relations—over the alternatives. On the other end of the spectrum, the definition also allows for *incomplete* or *quasi-transitive* preferences.[2]

Given a *preference profile* $(\succsim_i)_{i \in N}$, we say that alternative a *dominates* alternative b, in symbols $a > b$, whenever the number of voters i for which $a \succsim_i b$ exceeds the number of voters i for which $b \succsim_i a$. Obviously, the dominance relation is *asymmetric*. Despite the fact that most of the social choice literature has focused on *tournaments* (see *e.g.*, Laslier, 1997; Laffond et al., 1995; Moulin, 1986; Miller,

[1] Due to the possibility of ties, many common choice *rules* do not always select only one alternative. Thus the distinction between a choice rule and a choice *set*, as a criterion for a choice rule to select its alternatives from, is merely a gradual one.

[2] We say a relation $\geq$ is *asymmetric* whenever $x \geq y$ implies $y \ngeq x$. We say $\geq$ is *anti-symmetric* whenever $x \geq y$ and $y \geq x$ imply $x = y$. The relation $\geq$ is *quasi-transitive* if $>$ (the strict part of $\geq$) is transitive.

1977), *i.e.*, complete dominance relations, the dominance relation need not in general be *complete*.[3] In fact, McGarvey (1953) shows that *any* dominance relation can be realized by a particular preference profile for a number of voters polynomial in m, even if individual preferences are transitive, complete and anti-symmetric. In the presence of *incomplete* or *quasi-transitive* preferences, incomplete dominance relations are rather the norm than just a theoretical possibility. In the remainder of this paper, we will be mainly concerned with dominance relations and tacitly assume appropriate underlying individual preferences.

3 CHOICE SETS

In this section, we motivate and introduce six choice sets based on the pairwise majority dominance relation and analyze the relationships between these sets.

We say that an alternative $a \in A$ is *undominated* in $X \subseteq A$ relative to $\succ$, whenever there are no alternatives $b \in X$ with $b \succ a$. We say that an element is *undominated* if it is undominated in A. A special type of undominated alternative is the *Condorcet winner*, which is an alternative that dominates every other alternative and is dominated by none. The concept of a *maximal element* we reserve in this paper for transitive (and possibly reflexive) relations $\geq$. An alternative $a \in A$ is said to be *maximal* in such a transitive relation, if there is no $b \in A$ such that $b \geq a$ but not $a \geq b$. Equivalently, the maximal elements of $\geq$ can be defined as the undominated elements in the strict (*i.e.*, asymmetric) part of $\geq$.

Given its asymmetry, transitivity of the dominance relation implies its acyclicity. The implication in the other direction holds for tournaments but not for the more general case. Failure of transitivity or completeness makes that a Condorcet winner need not exist; failure of acyclicity, moreover, that the dominance relation need not even contain maximal elements. As such, the obvious notion of maximality is no longer available to single out the "best" alternatives among which the social choice should be selected. Other concepts had to be devised to take over its role. In this paper, we will be concerned with six of these concepts: the Copeland set, the Smith set, the Schwartz set, von Neumann-Morgenstern stable sets, the Banks set, and the Slater set.

3.1 DEFINITIONS

If a Condorcet winner exists, it is obviously the alternative that dominates the greatest number of alternatives, *viz.* all but itself, and is dominated by the smallest number, *viz.* by none. The *Copeland set* varies on this theme, by singling out those alternatives that maximize the difference between the number of alternatives they dominate and the number of alternatives they are dominated by (Copeland, 1951).

Definition 1 (Copeland score and Copeland set) *The* Copeland score $c(a)$ *of an alternative a given a dominance relation $\succ$ on a set of alternatives A equals* $|\{x \in A \mid a \succ x\}| - |\{x \in A \mid x \succ a\}|$. *The* Copeland set C *is given by* $\{a \in A \mid c(a) \geq c(x), \text{for all } x \in A\}$, i.e., *the set of alternatives with the maximal Copeland score.*

Obviously, the Copeland set never fails to be nonempty and contains the Condorcet winner as its only element if there is one.

A set of alternatives X has the *Smith property* if any alternative in X dominates any alternative not in X, *i.e.*, if $x \succ y$ holds for all $x \in X$ and all $y \notin X$. Note that the set of all alternatives satisfies this property, and hence the existence of at least one subset of alternatives with the Smith property is trivially guaranteed. The sets with the Smith property are, moreover, totally ordered by set inclusion. Hence, having assumed the set of alternatives to be finite, a unique *smallest* nonempty subset of alternatives with the Smith property cannot fail to exist. This set, as it was originally proposed by Smith (1973), we refer to as the *Smith set*.[4]

Definition 2 (Smith set) *The* Smith set S *is the smallest nonempty set of alternatives with the Smith property,* i.e., *such that $a \succ b$, for all $a \in S$ and all $b \notin S$.*

If the Smith set contains only one element, this alternative is the Condorcet winner. Numerous choice rules always pick alternatives from the Smith set, such as Nanson, Kemeny, or Fishburn (see, *e.g.*, Fishburn, 1977).

We say that a subset X of alternatives has the *Schwartz property* whenever no alternative in X is dominated by some alternative not in X, *i.e.*, for no $x \in X$ there is a $y \notin X$ with $y \succ x$. Vacuously the set of all alternatives satisfies the Schwartz property and so the existence of a nonempty subset with the Schwartz property is guaranteed. In contradistinction to the subsets with the Smith property, however, there need not be in general a *unique* minimal nonempty subset with the Schwartz property. With the set of alternatives having been assumed to be finite, we can single out those subsets with the Schwartz property that are both nonempty and are minimal ('smallest') with respect to set inclusion. We say that an alternative is in the *Schwartz set*, whenever it is an alternative of some such minimal subset with the Schwartz property (Schwartz, 1972).

Definition 3 (Schwartz set) *The* Schwartz set $T \subseteq A$ *is the union of all sets $T' \subseteq A$ such that:*

[3] Obviously, one is guaranteed to obtain a complete dominance relation if the number of voters is odd and individual preferences are linear.

[4] The Smith set appears in the literature under various names such as *top cycle* or *Condorcet set*. It is also sometimes confused with the Schwartz set (or *minimal undominated set*) because in *tournaments* both sets coincide.

(i) there is no $b \notin T'$ and no $a \in T'$ with $b > a$, and

(ii) there is no nonempty proper subset of T' that fulfills property (i).

Alternatively, the Schwartz set can be defined as the set of maximal elements of the transitive closure of the dominance relation (*cf.* Lemma 1). It is also worth observing that, if the dominance relation is acyclic, the Schwartz set consists precisely of all undominated alternatives. Moreover, unlike the Smith set (and stable sets below), the Schwartz set can contain a single alternative without this alternative being the Condorcet winner. If there is a Condorcet winner, however, it will invariably be the only element of the Schwartz set. The Schwartz set coincides with the Smith set if the dominance relation is complete, *i.e.*, in the case of tournaments. Well-known choice rules that always pick alternatives from the Schwartz set are Schulze and Ranked Pairs (see, *e.g.*, Schulze, 2003).

The intuition behind *stable sets* can perhaps best be understood by thinking of the social choice situation as one in which the voters have to settle upon a selection of alternatives from which the eventual social choice is to be selected by lot or some other mechanism beyond their control. One could argue that any such selection should at least satisfy two properties. No majority can be found in favor of restricting the selection by excluding some alternative from it. In a similar vein, it must be possible to find a majority against each proposal to include an outside alternative in the selection. Formally, stable sets are defined as follows.

Definition 4 (Stable set) *A set of alternatives $U \subseteq A$ is* stable *if it satisfies the following two properties, also known as* internal *and* external *stability, respectively:*

(i) $a > b$, for no $a, b \in U$, and

(ii) for all $a \notin U$ there is some $b \in U$ with $b > a$.

Equivalently, stable sets can be given a single fixed point characterization: the alternatives in a *stable* set U are precisely those that are undominated by any alternative in U. Observe that this definition does not exclude the possibility that an alternative outside a stable set dominates an alternative inside it.

Stable sets were proposed by von Neumann and Morgenstern (1944) to deal with intransitive dominance relations on imputations in the absence of a sensible concept of maximality. Originally, they were introduced as a solution concept for cooperative games and as such they have been studied extensively, especially in the 1950s. Richardson (1953), although also driven by game-theoretic motives, researched their formal properties in a more abstract setting. Within the context of social choice, stable sets have been paid considerably less attention to. If considered at all, it is only for a restricted class of situations (see, *e.g.*, Lahiri, 2004) or the concept is modified to some extent (see, *e.g.*, Dutta, 1988; van Deemen, 1991). One reason might be that in tournaments, a stable set exists if and only if there is a Condorcet winner, which it then contains as its only element. In the general case, however, neither uniqueness nor existence of stable sets is guaranteed. If the dominance relation is transitive, there is a unique stable set, which consists precisely of its maximal elements (and thus equals the Schwartz set). Moreover, a stable set is unique and a singleton if and only if there is Condorcet winner.

In contrast to the normative choice sets defined so far, Banks (1985) considers the set of possible winners of a pairwise elimination procedure if voters vote strategically. An alternative is in the *Banks set* if it is the maximal element of a subset of the alternatives for which the dominance relation is complete and transitive and which is itself maximal with respect to subset inclusion.

Definition 5 (Banks set) *An alternative $a_1 \in A$ is in the* Banks set *if there exists a subset $A' \subseteq A$, $A' = \{a_1, \ldots, a_k\}$, such that*

(i) $a_i > a_j$ for all $1 \leq i < j \leq k$, and

(ii) there is no $b \in A$ such that $b > a_i$ for all $1 \leq i \leq k$.

It is not very hard to show that the Banks set is a singleton if and only if there is a Condorcet winner.

The last set we consider is the Slater set which contains the maximal alternatives of those acyclic relations that disagree with the original dominance relation for a minimal number of tuples (Slater, 1961).

Definition 6 (Slater set) *An alternative $a \in A$ is in the* Slater set *if a is undominated in an acyclic subrelation of $>$ with maximal cardinality.*

The Slater set contains the Condorcet winner as its only element if it exists. However, the Slater set may also be a singleton in cases when no Condorcet winner exists. We conclude this section by stating without proof that none of the considered sets may contain the Condorcet loser, *i.e.*, an alternative that is dominated by all other alternatives.

3.2 DOMINANCE AND DIGRAPHS

It is very convenient to view the dominance relation derived from the voters' preferences as a directed graph $G = (V, E)$ where the set V of vertices equals the set A of alternatives and there is a directed edge $(a, b) \in E$ for $a, b \in V$ if and only if $a > b$ (see, *e.g.*, Miller, 1977). Figure 1 shows the digraph obtained for a set of six alternatives and the following profile of partial preferences for five voters (to improve readability, we only give the strict part of the preference

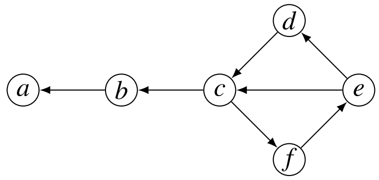

Figure 1: Dominance graph over a set of six alternatives and with Copeland set $C = \{e\}$, Smith set $S = \{a, b, c, d, e, f\}$, Schwartz set $T = \{c, d, e, f\}$, the unique stable set $U = \{b, d, f\}$, Banks set $B = \{b, c, e, f\}$, and Slater set $L = \{e, f\}$

ordering $\succsim_i$ for each voter $i \in N$):

$$e \succ_1 d \succ_1 c \succ_1 b \succ_1 a$$
$$b \succ_2 a \succ_2 e, d \succ_2 c \succ_2 f$$
$$a \succ_3 c, f \succ_3 e \succ_3 d$$
$$a \succ_4 c \succ_4 e, a \succ_4 b \succ_4 d$$
$$e \succ_5 c \succ_5 a$$

Since all choice sets considered in this paper are defined in terms of the dominance relation only, we will henceforth restrict our attention to dominance graphs. From a computational perspective, we merely make the assumption that determining the dominance relation from a preference profile is easy, *i.e.*, no harder than computing the majority function on a string of bits. This is a reasonable assumption, since hardness of this operation obviously would imply hardness of any choice rule that takes individual preferences into account.

3.3 RELATIONSHIPS BETWEEN CHOICE SETS

Laffond et al. (1995) have conducted a thorough comparison of choice sets and derived various inclusions. However, their study is restricted to tournaments (where many of the following observations are not possible because the Smith set and the Schwartz set coincide) and does not cover stable sets. For these reasons, this section provides an exhaustive set-theoretic analysis of the concepts defined in Section 3.1. We start by observing that all sets we consider are contained in the Smith set. Due to space restrictions, the proofs of the results in this section are omitted.

Theorem 1 *The Copeland set, the Schwartz set, every stable set, the Banks set, and the Slater set are contained in the Smith set.*

We leave it to the reader to verify that no other inclusion relationships between the discussed sets hold. In order to further investigate the relationships between the considered choice sets, we provide a useful alternative characterization of the Schwartz set.

Lemma 1 *An alternative $a \in A$ is in the Schwartz set if and only if for every $b \in A$ such that there is a path from b to a in the dominance graph, there also is a path from a to b.*

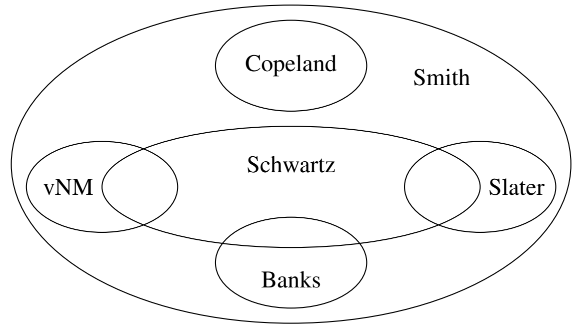

Figure 2: Relationships between the choice sets considered in this paper. Sets that intersect in the diagram *always* intersect. Sets that are disjoint in the diagram *may* have an empty intersection, *i.e.*, there exist instances where these sets do not intersect.

It turns out that all considered sets except the Copeland set always intersect with the Schwartz set.

Theorem 2 *Every stable set, the Banks set, and the Slater set always have a nonempty intersection with the Schwartz set.*

There exist no other intersections between the considered choice sets that are in general nonempty.

Theorem 3 *Any stable set, the Banks set, the Slater set, and the Copeland set may be pairwise disjoint.*

This completes our picture of inclusion and intersection relationships between the considered choice sets. Figure 2 combines all results of this section in one diagram.

4 COMPLEXITY RESULTS

In the remainder of the paper, we investigate the computational complexity of the considered choice sets. We start by defining decision problems for the Condorcet winner and each of the six choice sets defined in Section 3.1 as follows: given a set A of alternatives, a particular alternative $a \in A$, and a preference profile $(\succsim_i)_{i \in N}$, IS-CONDORCET asks whether alternative a is the Condorcet winner for preference profile $(\succsim_i)_{i \in N}$, and IN-COPELAND, IN-SMITH, IN-SCHWARTZ, IN-STABLE, IN-BANKS, and IN-SLATER ask whether a is contained in the Copeland set, the Smith set, the Schwartz set, a stable set, the Banks set, and the Slater set for $(\succsim_i)_{i \in N}$, respectively. We further assume the reader to be familiar with the well-known chain of complexity classes $AC^0 \subset TC^0 \subseteq L \subseteq NL \subseteq NC \subseteq P \subseteq NP$, and the notions of constant-depth and polynomial-time reducibility (see, *e.g.*, Johnson, 1990). AC^0 is the class of problems solvable by uniform constant-depth Boolean circuits with unbounded fan-in and a polynomial number of gates. TC^0 is defined by additionally allowing so-called threshold gates which yield *true* if and only if the number of *true* inputs exceeds a certain threshold. Basic functions computable in this class have been investigated by Chandra et al. (1984). We say that a problem is *complete* for TC^0 if it is complete under AC^0 Turing reductions. NC is the class of problems solvable by Boolean circuits with bounded fan-in and a polynomial number of gates. L and NL are the classes of problems solvable by deterministic and nondeterministic Turing machines using only logarithmic space, respectively. P and NP are the classes of problems that can be solved in polynomial time by deterministic and nondeterministic Turing machines, respectively.

First, we observe that a particular entry in the adjacency matrix of the dominance graph for a preference profile $(\succsim_i)_{i \in N}$ is given by the majority function for a particular pair of alternatives, and that the complete adjacency matrix can be computed in TC^0. Showing that IS-CONDORCET is in TC^0 is also straightforward. We just have to check whether all off-diagonal entries in the row of the adjacency matrix corresponding to a are 1. Hardness, on the other hand, follows from the fact that the case with two alternatives is equivalent to computing the majority function on a string of bits, which in turn is hard for TC^0. For IN-COPELAND, we have to check whether the difference between outdegree and indegree of the vertex corresponding to a is maximal over all vertices in the dominance graph. We can do this by computing, for each row of the adjacency matrix in parallel, the sum of all entries in this row and subtract the sum of all entries in the corresponding column. Finally, we check whether the result for the row (and column) corresponding to a attains the maximum over all pairs of rows (and corresponding columns). Hardness follows from the fact that IN-COPELAND and IS-CONDORCET are equivalent for the case of two alternatives and an odd number of voters with linear preferences.

It is well-known that both the Smith set and the Schwartz set can be computed in polynomial time by applying the algorithm of Kosaraju for finding strongly connected components in the dominance graph (see, *e.g.*, Cormen et al., 2001). In graph-theoretic terms, the Smith set is the highest (with respect to the dominance relation) strongly connected component in the digraph for the *majority-or-tie* dominance relation, while the Schwartz set is the highest strongly connected component for the *majority* dominance relation. Our approach for computing the Smith set is quite different and based on the in- and outdegree of vertices inside and outside that set. Assume there exists a Smith set $S \subseteq A$ of size k. Since by definition every member of S must dominate every non-member, the outdegree of every element of S in the dominance graph for A must be at least $n - k$, while every alternative not in S must have indegree at least k. Furthermore, no alternative can satisfy both properties because the sum of in- and outdegree for each vertex in an asymmetric digraph is bounded by $n - 1$. Given a particular k, we can thus try to partition A into two sets S' and $\bar{S}' = A \setminus S'$ by the above criterion, such that S' is the unique candidate for a set of size k that satisfies the Smith property. We can then easily check whether S' actually satisfies the Smith property, and find the Smith set by performing this process for $1 \leq k \leq n$ in parallel. We proceed to show that this algorithm can be implemented using a constant-depth threshold circuit, and that checking membership in the Smith set is actually complete for the class TC^0.

Theorem 4 *IN-SMITH is TC^0-complete (under AC^0 Turing reductions).*

Proof: Hardness is immediate from the equivalence of IN-SMITH and IS-CONDORCET for the case of two alternatives and an odd number of voters with linear preferences.

For *membership*, we construct a constant-depth threshold circuit that decides whether there exists a set of size k with the Smith property. We can then perform the checks for all possible values of k in parallel, and decide whether a particular alternative is in the smallest such set. We start by computing the adjacency matrix $M = (m_{ij})$ of the dominance graph from the preference profile. This amounts to a polynomial number of majority votes over pairs of alternatives and can obviously be done in TC^0. We then apply a threshold of $n - k$ to each row of M to obtain a vector v such that v_i is *true* if and only if the ith alternative is in the potential Smith set S'. To decide whether S' actually satisfies the Smith property, we have to check whether the outdegree of vertices in S' is still high enough if we only consider edges to vertices in $\bar{S}'$, *i.e.*, whether the properties regarding in- and outdegree are satisfied for the *bipartite part* of A with respect to S' and $\bar{S}'$. We thus compute the adjacency matrix $M^b = (m^b_{ij})$ for the bipartite part of A as $m^b_{ij} = (m_{ij} \wedge \neg v_j)$ and again apply a threshold of $n - k$ to each row to yield a vector v^b. S' satisfies the Smith property if and only if a threshold of k applied to v^b yields *true*. In this case, the ith alternative is contained in this set if $v^b_i = true$. □

The previous theorem implies that any choice rule that picks its winner from the Smith set is TC^0-hard, and thus in principle not harder than any Condorcet choice rule. As noted above, the Smith set and the Schwartz set differ only by their treatment of ties in the pairwise comparison. Nevertheless, and quite surprisingly, deciding membership in the Schwartz set is computationally harder unless TC^0=NL.

Theorem 5 *IN-SCHWARTZ is NL-complete (under AC^0 many-one reductions).*

Proof: Given a dominance graph and using Lemma 1, membership of an alternative $a \in A$ in the Schwartz set can be shown by checking for every other alternative $b \in A$ that either b is reachable from a or a is not reachable from b. Clearly, the existence of a particular edge in the dominance graph and hence the existence of a path between a pair of vertices can be decided by a nondeterministic Turing machine using only logarithmic space. Membership in the Schwartz set can then be decided using an additional pointer into the input to store alternative b.

For *hardness*, we provide a reduction from the NL-complete problem of digraph reachability (see, *e.g.*, Johnson, 1990) which obviously is still NL-complete even when the graph does not contain double edges. Given a particular digraph $G = (V, E)$ and two designated vertices $s, t \in V$, we construct a dominance graph $G' = (V', E')$ by adding two additional vertices s' and t', an edge from s' to s, an edge from t to t', edges from any vertex but s to s', and edges from t' to any vertex but t, *i.e.*,

$$\begin{aligned} V' &= V \cup \{s', t'\} \quad \text{and} \\ E' &= E \cup \{(s', s)\} \cup \{ (v, s') \mid v \in V, v \neq s \} \\ &\quad \cup \{(t, t')\} \cup \{ (t', v) \mid v \in V, v \neq t \}. \end{aligned}$$

It is easily verified that G' can be computed from G by a Boolean circuit of constant depth. We claim that s is contained in the Schwartz set for G' if and only if there exists a path from s to t in G. First of all, we observe that a path from s to t in G' exists if and only if such a path already existed in G, since we have not added any outgoing edges to s or any incoming edges to t. By construction, every vertex of G', including s, can be reached from t. Hence, by Lemma 1, s cannot be contained in the Schwartz set if t cannot be reached from s. Conversely assume that t is reachable from s. Then this property holds for every vertex of G' as well, particularly for those from which s can be reached. In virtue of Lemma 1, we may conclude that s is in the Schwartz set. Furthermore, since s is reachable from every vertex (via s'), *all* vertices are contained in the Schwartz set if and only if there is path from s to t. □

Naturally, hardness of the membership decision problem for a particular set does not automatically imply hardness of all choice rules that always yield an alternative from that set. For example, we will see later that finding an arbitrary alternative from the Banks set is actually *easier* than deciding whether a given alternative is contained in it. However, in the case of the Schwartz set, we can prove the hardness of all *Schwartz-consistent* choice rules, *i.e.*, choice rules that always select an alternative from the Schwartz set, under a mild tie-breaking condition. Consider a dominance relation with several minimal sets that satisfy the Schwartz property. A choice rule with *fixed tie-breaking order* may arbitrarily pre-select a "candidate" from each of these sets. However, which alternative is ultimately chosen from these candidates only depends on a predefined order that is independent of the voters' preferences. Thus, we may assume that alternatives possess consecutive indices and that an alternative may only be chosen if every other minimal set satisfying the Schwartz property contains an alternative with a higher index.

Proposition 1 *Consider a choice rule that selects an alternative from the Schwartz set using a fixed tie-breaking order. This choice rule cannot be executed on a deterministic Turing machine with logarithmic space unless L = NL.*

Proof: As pointed out in the proof of Theorem 5, the problem of deciding whether *all* alternatives are contained in the Schwartz set is NL-hard. This can be used to show via an AC^0 Turing reduction that deciding SCHWARTZ-SINGLETON, *i.e.*, deciding whether the Schwartz set contains only one element, is also NL-hard. More precisely, we provide a constant-depth unbounded fan-in circuit with access to a SCHWARTZ-SINGLETON oracle that for a given dominance graph $G = (V, E)$ decides whether the Schwartz set contains all vertices. For every vertex $v \in V$, we let the oracle decide SCHWARTZ-SINGLETON for $G_v = (V \cup u, E \cup \{u, v\})$. The Schwartz set of G contains all vertices if and only if the oracle yields a positive answer for every G_v.

We proceed to show the hardness of every Schwartz-consistent choice rule with a fixed tie-breaking order using an AC^0 Turing reduction from SCHWARTZ-SINGLETON. Let f be a Schwartz-consistent choice rule, $G = (V, E)$ be an arbitrary dominance graph, and v be the alternative that f yields for graph G. If v's indegree is greater than zero, we can easily decide SCHWARTZ-SINGELTON because there has to be another vertex in the Schwartz set. If, on the other hand, v is undominated, we run f on the modified graph $G' = (V \cup u, E \cup \{u, v\})$ and let u's tie-breaking index be greater than all existing indices. If f still yields v, there cannot be another minimal set with the Schwartz property which implies that v is the only element of the Schwartz set. Whenever there exists another minimal Schwartz set, f must yield a vertex different from v due to a lower tie-breaking index and SCHWARTZ-SINGELTON can be decided in the negative. □

The previous proposition can be used to show that well-known Schwartz-consistent choice rules like Ranked Pairs or Schulze are NL-hard (given a fixed tie-breaking order). For all choice sets considered so far, we can check efficiently whether they contain a particular alternative or not. Unfortunately, this is not case for stable sets (unless P=NP).

Theorem 6 *IN-STABLE is NP-complete, even if a nonempty stable set is guaranteed to exist.*

Proof: Membership in NP is obvious. Given a dominance graph over a set A of alternatives and a particular alternative $a \in A$, we can simply guess a subset $U \subseteq A$ such that $a \in U$, and verify that for every $b \notin U$ there is an edge from some element of U to b and that there are no edges between vertices of U.

For *hardness*, we provide a reduction from satisfiability of a Boolean formula B (SAT) to the problem of deciding whether a designated alternative $a \in A$ is contained in a stable set (or the union of all stable sets). The reduction is based on the reduction by Chvátal (1973) to show NP-hardness of the problem of deciding whether a digraph has a kernel. Let $B = \bigwedge_{1 \leq i \leq m} \bigvee_{1 \leq j \leq k_i} p_{ij}$ be a SAT instance over variables X. We construct an asymmetric dominance graph $G = (V, E)$ with three vertices c_{i1}, c_{i2}, and c_{i3} for each clause of B, four vertices x_i, $\bar{x}_i$, x'_i, and $\bar{x}'_i$ for each variable of B, and four additional vertices d_1, d_2, d_3, and d_4, such that d_1 is contained in a stable set if and only if B has a satisfying assignment. Vertices c_{ij} will henceforth be called clause vertices, x_i and $\bar{x}_i$ will be referred to as positive and negative literal vertices, respectively. Edges are such that the vertices of each clause form a directed cycle of length three, and the vertices of each variable as well as the decision vertices form a cycle of length four according to the sequences given above. Furthermore, there is an edge from a positive or negative literal vertex to all clause vertices of a clause in which the respective literal appears. Finally, there is an edge from d_2 to every clause vertex. More formally, we have

$$
\begin{aligned}
E = \{ &(d_1, d_2), (d_2, d_3), (d_3, d_4), (d_4, d_1) \} \cup \\
&\{ (c_{i1}, c_{i2}), (c_{i2}, c_{i3}), (c_{i3}, c_{i1}) \mid 1 \leq i \leq m \} \cup \\
&\{ (x_i, \bar{x}_i), (\bar{x}_i, x'_i), (x'_i, \bar{x}'_i), (\bar{x}'_i, x_i) \mid 1 \leq i \leq |X| \} \cup \\
&\textstyle\bigcup_{1 \leq \ell \leq k_j} \{ (x_i, c_{j1}), (x_i, c_{j2}), (x_i, c_{j3}) \mid p_{j\ell} = x_i \} \cup \\
&\textstyle\bigcup_{1 \leq \ell \leq k_j} \{ (\bar{x}_i, c_{j1}), (\bar{x}_i, c_{j2}), (\bar{x}_i, c_{j3}) \mid p_{j\ell} = \bar{x}_i \} \cup \\
&\{ (d_2, c_{i1}), (d_2, c_{i2}), (d_2, c_{i3}) \mid 1 \leq i \leq m \}.
\end{aligned}
$$

Figure 3 illustrates this construction for a particular Boolean formula. We observe the following facts:

- G can be constructed from B in polynomial time.
- $\{ x_i, x'_i \mid 1 \leq i \leq m \} \cup \{d_2, d_4\}$ is a stable set of G irrespective of the structure of B.
- Every stable set of G must either contain d_1 and d_3 or d_2 and d_4, but not both. For each i, every stable set must either contain x_i and x'_i or $\bar{x}_i$ and $\bar{x}'_i$, but not both.
- A stable set of G cannot contain a pair of clause vertices for the same clause. In turn, a stable set must contain vertices with outgoing edges to at least two of the three vertices for every clause. However, every vertex that has an outgoing edge to any vertex for some clause has an outgoing vertex to all three vertices for that clause. Hence, a stable set cannot contain any clause vertices.
- A stable set must contain either d_2 or a subset of the literal vertices containing at least one vertex for a literal in every clause. Since a stable set cannot contain both x_i and $\bar{x}_i$, the latter corresponds to a satisfying assignment B. Hence, a stable set containing d_1 exists if and only if B is satisfiable.

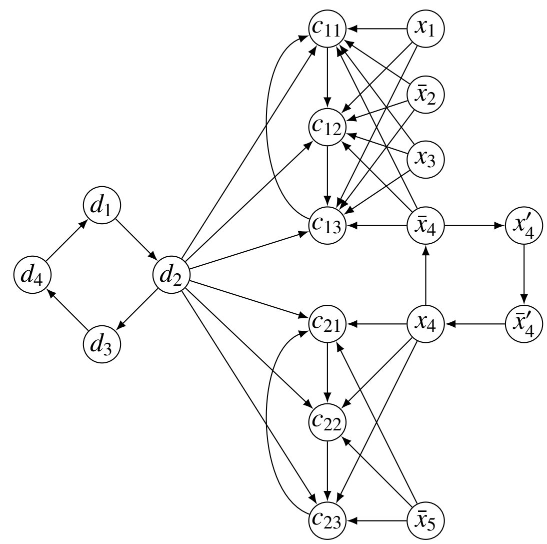

Figure 3: Dominance graph for the Boolean formula $(x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_4 \vee \bar{x}_5)$ according to the construction used in the proof of Theorem 6. If a certain variable appears exclusively as either a positive or negative literal, the other three vertices for the variable are omitted.

This completes the proof. □

As in the case of the Schwartz set, we can derive a stronger result, concerning the computational complexity of *any* choice rule that is guaranteed to select an alternative from a stable set, if such an alternative exists.

Proposition 2 *Consider a choice rule that selects an alternative from a stable set if one exists and an arbitrary alternative otherwise. This choice rule cannot be executed in worst-case polynomial time unless P=NP.*

Proof: Again consider the construction used in the proof of Theorem 6 and illustrated in Figure 3. In this construction, four designated vertices d_1 to d_4 have been used to guarantee the existence of a stable set, no matter whether the underlying Boolean formula B has a satisfying assignment or not. This guarantee also means that finding *some* alternative that belongs to a stable set is trivial. It is easily verified that if we remove vertices d_1 to d_4, a stable set in graph G exists if and only if B has a satisfying assignment, and the vertices in such a stable set are those corresponding to the literals set to true in a particular satisfying assignments.

Now consider a Turing machine with an oracle that computes a single alternative belonging to a stable set, if such a set exists, and an arbitrary alternative otherwise. Using this machine, the existence of a satisfying assignment for a particular Boolean formula B can be decided as follows. First, compute the dominance graph $G = (V, E)$ corresponding to B. Then, iteratively reduce the graph by requesting a vertex v from the oracle and removing vertices as follows: if $v = x_i$ or $v = x'_i$ for some $1 \le i \le |X|$, remove x_i, x'_i, $\bar{x}_i$, $\bar{x}'_i$ and all c_{ij} such that $(x_i, c_{ij}) \in E$; if $v = \bar{x}_i$ or $v = \bar{x}'_i$ for some $1 \le i \le |X|$, remove x_i, x'_i, $\bar{x}_i$, $\bar{x}'_i$ and all c_{ij} such that $(\bar{x}_i, c_{ij}) \in E$. If at some point there no longer exists any vertex c_{ij}, let the machine halt and accept. If at some point there no longer exists any x_i or $\bar{x}_i$ but there still is some c_{ij}, or if the oracle returns c_{ij} for some $1 \le i \le m$, $j \in \{1, 2, 3\}$, let the machine halt and reject.

As already pointed out in the proof of Theorem 6, the graph G can be computed from B in polynomial time. In every later step, the machine either halts or removes at least one vertex, of which there are only polynomially many. Hence, the machine is guaranteed to halt after a polynomial number of steps. Furthermore, if the machine accepts, the set of all vertices returned by the oracle form a stable set of G, which can only exist if B has a satisfying assignment. We have thus provided a Turing reduction from SAT to the problem of selecting an arbitrary element of a stable set, showing that a polynomial-time algorithm for the latter would imply P=NP. □

While the union of all stable sets need not in general be contained in the Schwartz set (see *e.g.*, Figure 1), this is the case for the dominance graphs used in the proofs of the previous theorem. Hence, hardness holds as well for deciding whether an alternative lies in the intersection of a stable set and the Schwartz set, and for any choice rule that selects an alternative that is both in a stable set and in the Schwartz set.

For the Banks and the Slater set, the complexity of the membership decision problem as well as the corresponding search problems are well-studied. We just briefly list the results for completeness. Woeginger (2003) has shown that deciding whether a given alternative is contained in the Banks set of a tournament is NP-complete via a reduction from graph 3-colorability. Interestingly, an arbitrary element of the Banks set can be found in polynomial time (Hudry, 2004). The membership decision as well as the search problem of the Slater set are long known to be NP-hard since they are equivalent to the classic NP-complete problem *minimum feedback arc set*. It was recently proven that hardness holds even when the graph is a tournament (Alon, 2006).

5 CONCLUSION

We have investigated the relationships and computational complexity of various choice sets based on the pairwise majority relation. Table 1 summarizes the complexity-theoretic results, which can be interpreted as follows. All considered problems except IN-STABLE, IN-BANKS, and IN-SLATER are computationally tractable. Moreover, these problems are contained in the complexity class NC of problems amenable to parallel computation. All problems except IN-SCHWARTZ, IN-STABLE, IN-BANKS, and IN-SLATER can be solved on a deterministic Turing machine using only logarithmic space. These results can be used to make statements regarding the complexity of entire classes of choice rules, *e.g.*, the NL-hardness of every choice rule that picks an alternative from the Schwartz set or the NP-hardness of every choice rule that picks an alternative from a stable set.

In addition, Table 1 underlines the significant difference between tournaments and general dominance graphs. Surprisingly, the Smith set turned out to be computationally easier than the Schwartz set in general dominance graphs (unless TC^0=NL), while both concepts coincide in tournaments. Deciding whether an alternative is included in a stable set is NP-complete in general dominance graphs, while in tournaments the same problem is equivalent to the TC^0-complete problem of deciding whether the alternative is the Condorcet winner.

Finally, it should be noted that our results are fairly general in the sense that they rely only on the *asymmetry* of the dominance relation. As a matter of fact, all considered sets are reasonable substitutes for maximality in the face of non-transitive relations, no matter whether these relations stem from aggregated preferences or not.

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<table>
<tr><th rowspan="2">Choice set</th><th colspan="2">Membership decision problem in</th><th rowspan="2">Hardness of corresponding choice rules</th></tr>
<tr><th>tournaments</th><th>general dominance graphs</th></tr>
<tr><td>Condorcet</td><td rowspan="5">TC^0-complete</td><td rowspan="3">TC^0-complete</td><td rowspan="3">TC^0-hard</td></tr>
<tr><td>Copeland</td></tr>
<tr><td>Smith</td></tr>
<tr><td>Schwartz</td><td>NL-complete</td><td>NL-hard[a]</td></tr>
<tr><td>vNM</td><td>NP-complete</td><td>NP-hard</td></tr>
<tr><td>Banks</td><td colspan="2">NP-complete[b]</td><td>at most P-hard[c]</td></tr>
<tr><td>Slater</td><td colspan="2">NP-complete[d]</td><td>NP-hard</td></tr>
</table>

[a]for fixed tie-breaking order, see Proposition 1

[b]Woeginger (2003)

[c]Hudry (2004) pointed out that an arbitrary element of the Banks set can be found in polynomial time, which implies that choice rules choosing from the Banks set cannot be harder than P-hard.

[d]Alon (2006)

Table 1: Computational complexity of choice sets

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