Full and Relative Awareness: A Decidable Logic for Reasoning about Knowledge of Unawareness

Thomas Ågotnes
Bergen University College
Norway
tag@hib.no

Natasha Alechina
University of Nottingham
UK
nza@cs.nott.ac.uk

Abstract

In the most popular logics combining knowledge and awareness, it is not possible to express statements about knowledge of *un*awareness such as "Ann knows that Bill is aware of something Ann is not aware of" – without using a stronger statement such as "Ann knows that Bill is aware of p and Ann is not aware of p", for some particular p. Recently, however, Halpern and Rêgo (2006) introduced a logic in which such statements about knowledge of unawareness can be expressed. The logic extends the traditional framework with quantification over formulae, and is thus very expressive. As a consequence, it is not decidable. In this paper we introduce a decidable logic which can be used to reason about certain types of unawareness. The logic extends the traditional framework with an operator expressing *full awareness*, i.e., the fact that an agent is aware of everything, and another operator expressing *relative awareness*, the fact that one agent is aware of everything another agent is aware of. The logic is less expressive than Halpern's and Rêgo's logic. It is, however, expressive enough to express all of Halpern's and Rêgo's motivating examples. In addition to proving that the logic is decidable and that its satisfiability problem is PSPACE-complete, we present an axiomatisation which we show is sound and complete.

1 Introduction

Formal models of knowledge or belief extended with a notion of *awareness* has been of interest to researchers in several fields, including economics and game theory, philosophy, and multi-agent systems. One of the most popular frameworks is the *logic of general awareness* (Fagin and Halpern, 1988), which has been shown (Halpern, 2001) to be a generalisation of frameworks used by economists (Modica and Rustichini, 1994, 1999). The logic of general awareness has a traditional (implicit) knowledge operator K_i where $K_i\phi$ is interpreted as truth of ϕ in all accessible worlds in a Kripke structure, in addition to an awareness operator A_i where $A_i\phi$ is interpreted by a syntactic assignment of truth value, and an explicit knowledge operator X_i such that $X_i\phi$ is interpreted as the conjunction of $K_i\phi$ and $A_i\phi$. This framework is very flexible and general. However, as recently pointed out by Halpern and Rêgo (2006), in many situations, agents have *knowledge about their own or others' unawareness*, and this cannot be expressed properly in the logic of general awareness. An example, taken directly from (Halpern and Rêgo, 2006), is the following.

Example 1

Consider an investor (agent 1) and an investment fund broker (agent 2). Suppose that we have two facts that are relevant for describing the situation: the NASDAQ index is more likely to increase than to decrease tomorrow (p), and Amazon will announce a huge increase in earnings tomorrow (q). [...] [B]oth agents explicitly know that the NASDAQ index is more likely to increase than to decrease tomorrow. However, the broker also explicitly knows that Amazon will announce a huge increase in earnings tomorrow. Furthermore, the broker explicitly knows that he (broker) is aware of this fact and the investor is not. On the other hand, the investor explicitly knows that there is something that the broker is aware of but he is not.

In order to be able to reason formally about situations involving knowledge of unawareness such as this one, Halpern and Rêgo (2006) introduced a logic which extends the logic of general awareness with variables standing for formulae and quantification over these variables. For example, the formula $X_1(\exists x(A_2x \wedge$

$\neg A_1 x$)) expresses the fact that the investor, in the example above, explicitly knows that there is some fact he is unaware of but the broker is aware of. This introduction of quantifiers makes the logic very expressive, but unfortunately also makes it undecidable.

There is a subtle distinction in the motivating arguments of Halpern and Rêgo (2006). On the one hand, it is initially argued that it would be useful to express the fact that an agent "knows that there are facts of which he is unaware". We will refer to awareness of *everything* as *full awareness*. Explicit knowledge of the lack of full awareness can be expressed in Halpern's and Rêgo's logic by a formula such as $X_i(\exists x \neg A_i x)$. On the other hand, Example 1 above requires the expression of knowledge of a more specific property of unawareness: that an agent (explicitly) knows that he is unaware of some fact which another agent *is* aware of. We will refer to this latter form of unawareness as lack of *relative awareness*. We say that an agent has relative awareness with respect to another agent if he is aware of everything the other agent is aware of. As discussed above, knowledge of lack of relative awareness can be expressed in Halpern's and Rêgo's logic by a formula such as $X_i(\exists x(A_j x \wedge \neg A_i x))$. Full awareness implies relative awareness, but in general not the other way around.

In this paper, we introduce an alternative logic for reasoning about knowledge of unawareness, which extends the logic of general awareness with explicit operators for full and relative awareness. For each agent i, the logic has a nullary operator C_i standing for "agent i has full awareness", and for each agent i and each agent j a nullary operator R_{ij} standing for for "agent j has relative awareness wrt. agent i". In this language, both types of knowledge of unawareness mentioned above can be expressed, viz. as $X_i \neg C_i$ and $X_i \neg R_{ji}$, respectively.

With these operators in place of unlimited quantification over formulae, the logic is, obviously, much less expressive than Halpern's and Rêgo's logic. However, it can be used to express all the motivating examples in (Halpern and Rêgo, 2006). Furthermore, the logic presented in this paper is decidable.

Of related work, both Modica and Rustichini (1999) and Halpern (2001) develop logics of unawareness, but for the single-agent case only. Board and Chung (2006) add awareness operators to first order logic. Sillari (2006) also combines first-order logic and awareness, this time interpreted over neighborhood structures. There is a fundamental difference, however, between quantification in these two latter frameworks and in that of Halpern and Rêgo (2006). In (Board and Chung, 2006) and (Sillari, 2006), quantification is over *objects* of the universe of discourse, while in (Halpern and Rêgo, 2006) quantification is over *formulae*. In general, we need the latter type of quantification to reason about unawareness of formulae. Heifetz et al. (2007) develop a set theoretic, as opposed to the syntactic approach of Halpern and Rêgo (2006), framework for unawareness.

This paper is organised as follows. In the next section we introduce the logic of general awareness, and Halpern's and Rêgo's logic. Our logic of full and relative awareness is then presented in Section 3, and an axiomatisation proved sound and (weakly) complete in Section 4. In Section 5 we compare the logic to Halpern's and Rêgo's logic. The satisfiability problem for the logic is studied in Section 6. We prove that the problem is decidable, and that it is PSPACE-complete. We conclude in Section 7.

2 Background: Logics of Awareness and Unawareness

In this paper we consider several logical languages $\mathcal{L}$. We define the meaning of each of these by defining the concept of a formula $\phi \in \mathcal{L}$ being true (or satisfied) in the context of the combination of a model $M \in \mathcal{M}$ in some class of models $\mathcal{M}$ and a state s of M, written $(M, s) \models \phi$. ϕ is valid (wrt. $\mathcal{M}$), written $\models \phi$, if $(M, s) \models \phi$ for all $M \in \mathcal{M}$ and all states s in M. We also consider (Hilbert style) logical systems S over $\mathcal{L}$; $\vdash_S \phi$ means that ϕ is derivable in S. S is sound wrt. $\mathcal{M}$ iff $\vdash_S \phi$ implies that $\models \phi$; S is complete if the converse holds.

2.1 Awareness Structures and The Logic of General Awareness

We briefly recall the logic of general awareness (Fagin and Halpern, 1988) (our notation is similar to that of Halpern and Rêgo (2006)).

An *awareness structure* for n agents over primitive propositions Φ and logical language $\mathcal{L}$ is a tuple $(S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n)$, where S is a non-empty set of states, $\pi : S \to \Phi$ says which primitive propositions are true in each state, $\mathcal{K}_i \subseteq S \times S$ is the accessibility relation for agent i, and $\mathcal{A}_i : S \to 2^{\mathcal{L}}$ defines the awareness set $\mathcal{A}_i(s) \subseteq \mathcal{L}$ for each agent i in each state $s \in S$. Intuitively, $(s, t) \in \mathcal{K}_i$ means that when the state of the world actually is s agent i considers it possible that the state of the world is t; $\phi \in \mathcal{A}_i(s)$ means that agent i is aware of the formula ϕ when the state of the world is s.

We shall consider several model classes, defined by requiring the accessibility relations to be reflexive ($(s, s) \in \mathcal{K}_i$ for all $s \in S$), transitive ($(s, t) \in \mathcal{K}_i$

and $(t,u) \in \mathcal{K}_i$ implies that $(s,u) \in \mathcal{K}_i$) and/or Euclidian ($(s,t) \in \mathcal{K}_i$ and $(s,u) \in \mathcal{K}_i$ implies that $(t,u) \in \mathcal{K}_i$). For $C \subseteq \{r,t,e\}$, we use $\mathcal{M}_n^C(\Phi,\mathcal{L})$ to denote the awareness structures for n agents over Φ and $\mathcal{L}$ where the accessibility relations are required to have the properties in C ("r" means reflexive, etc.). We sometimes write $\mathcal{M}_n(\Phi,\mathcal{L})$ for $\mathcal{M}_n^{\emptyset}(\Phi,\mathcal{L})$ – the class of all awareness structures.

Given a number n of agents and a set Φ of primitive propositions, the formulae ϕ of the language $\mathcal{L}_n^{K,X,A}(\Phi)$ are defined by the following grammar:

$$\phi ::= p \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid K_i\phi \mid X_i\phi \mid A_i\phi$$

where $p \in \Phi$ and $1 \leq i \leq n$. The usual derived propositional connectives are used, for example we write $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$ and so on. The formula $A_i\phi$ means that agent i is aware of ϕ.

Below we describe how awareness structures for n agents over primitive propositions Φ and logical language $\mathcal{L}_n^{K,X,A}(\Phi)$ are used to interpret the language $\mathcal{L}_n^{K,X,A}(\Phi)$. In the following sections of the paper we shall also look at other languages $\mathcal{L}$, and we will then use awareness structures for n agents over Φ and $\mathcal{L}$ to interpret $\mathcal{L}$.

The notion of a formula $\phi \in \mathcal{L}_n^{K,X,A}(\Phi)$ being true, or satisfied, in a state s of an awareness structure $M = (S,\pi,\mathcal{K}_1,\ldots,\mathcal{K}_n,\mathcal{A}_1,\ldots,\mathcal{A}_n) \in \mathcal{M}_n(\Phi,\mathcal{L}_n^{K,X,A}(\Phi))$, written $(M,s) \models \phi$, is defined as follows, where $p \in \Phi$ and $1 \leq i \leq n$:

$$\begin{array}{lcl}
(M,s) \models p & \Leftrightarrow & p \in \pi(s) \\
(M,s) \models \phi_1 \wedge \phi_2 & \Leftrightarrow & (M,s) \models \phi_1 \text{ and } (M,s) \models \phi_2 \\
(M,s) \models \neg\phi & \Leftrightarrow & (M,s) \not\models \phi \\
(M,s) \models K_i\phi & \Leftrightarrow & \forall (s,t) \in \mathcal{K}_i, (M,t) \models \phi \\
(M,s) \models A_i\phi & \Leftrightarrow & \phi \in \mathcal{A}_i(s) \\
(M,s) \models X_i\phi & \Leftrightarrow & (M,s) \models K_i\phi \wedge A_i\phi
\end{array}$$

Example 2 (Example 1 continued) *(Adapted from Halpern and Rêgo (2006)). The situation described in Example 1 up until immediately before the last sentence ("On the other hand..") can be modelled by an awareness structure* $M_2 = (S,\pi,\mathcal{K}_1,\mathcal{K}_2,\mathcal{A}_1,\mathcal{A}_2)$ *for 2 agents over the set* $\{p,q\}$ *of primitive propositions and logical language* $\mathcal{L}_2^{K,X,A}(\{p,q\})$*, defined as follows.* $S = \{s\}$*;* $\pi(s) = \{p,q\}$*;* $\mathcal{K}_1 = \mathcal{K}_2 = \{(s,s)\}$*;* $\mathcal{A}_1(s) = \{p\}$*;* $\mathcal{A}_2(s) = \{p,q,A_2q,\neg A_1q,A_2q \wedge \neg A_1q\}$*. The following hold:*

- $(M_2,s) \models X_1p \wedge X_2p$*: both the investor and the broker explicitly know that the NASDAQ index is more likely to increase than to decrease tomorrow*
- $(M_2,s) \models \neg X_1q \wedge X_2q$*: the investor does not explicitly know that Amazon will announce a huge increase in earnings tomorrow, but the broker does*
- $(M_2,s) \models X_2(A_2q \wedge \neg A_1q)$*: the broker explicitly knows that he (broker) is aware of this fact (regarding Amazon) and the investor is not*

2.2 A Logic of Knowledge of Unawareness

Halpern and Rêgo (2006) extended the logic of general awareness in order to be able to reason about knowledge of unawareness. We give a brief review of their logic, henceforth called *the HR logic*.

Let $\mathcal{X}$ be a countably infinite set of variables. The language extends the language of the logic of general awareness with variables, and formulae of the form $\forall x\phi$, where x is a variable. Formulas of $\mathcal{L}_n^{\forall,K,X,A}(\Phi,\mathcal{X})$ is defined by the following grammar:

$$\phi ::= p \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid K_i\phi \mid X_i\phi \mid A_i\phi \mid \forall x\phi \mid x$$

where $p \in \Phi$, $1 \leq i \leq n$ and $x \in \mathcal{X}$. We use the usual abbreviations in addition to $\exists x\phi$ for $\neg\forall x\phi$. A *sentence* is a formula without free variables; $\mathcal{S}_n^{\forall,K,X,A}(\Phi,\mathcal{X})$ denotes the set of all sentences.

Satisfaction of a $\mathcal{L}_n^{\forall,K,X,A}(\Phi,\mathcal{X})$ formula is defined in relation to a tuple consisting of an awareness structure $M \in \mathcal{M}_n(\Phi,\mathcal{S}_n^{\forall,K,X,A}(\Phi,\mathcal{X}))$, a state s in M, and a *syntactic valuation* $\mathcal{V} : \mathcal{X} \to \mathcal{L}_n^{K,X,A}(\Phi)$[1]. Note that awareness sets only contain sentences of $\mathcal{L}_n^{\forall,K,X,A}$. The definition of satisfaction is by nested induction, first over the total number of free and bound variables and then on the length of the formula. In addition to the clauses which appear in the definition of satisfaction for $\mathcal{L}_n^{K,X,A}(\Phi)$, the following two are used. We refer to Halpern and Rêgo (2006) for a more thorough explanation of the definition.

$$\begin{array}{l}
(M,s,\mathcal{V}) \models \phi \Leftrightarrow \\
\qquad (M,s,\mathcal{V}) \models \phi[x_1/\mathcal{V}(x_1),\ldots,x_k/\mathcal{V}(x_k)] \\
\qquad \text{when } \mathit{Free}(\phi) = \{x_1,\ldots,x_k\} \\
(M,s,\mathcal{V}) \models \forall x\phi \Leftrightarrow \\
\qquad (M,s,\mathcal{V}') \models \phi \\
\qquad \forall \mathcal{V}' \text{ such that } \mathcal{V}' \sim_x \mathcal{V}
\end{array}$$

where $\mathit{Free}(\phi) \subset \mathcal{X}$ denotes the set of free[2] variables (not bound by a quantifier) occurring in ϕ; $\phi[x_1/\mathcal{V}(x_1), \ldots, x_k/\mathcal{V}(x_k)]$ means the formula resulting from replacing in ϕ all free occurrences of x_j with the formula $\mathcal{V}(x_j)$ (for each j); and $\mathcal{V}' \sim_x \mathcal{V}$ means that $\mathcal{V}'(y) = \mathcal{V}(y)$ for every $y \in \mathcal{X} \setminus \{x\}$. For more details regarding this semantics and why it is well defined, we again refer to Halpern and Rêgo (2006).

[1] Note that in this framework, the interpretation of a variable is restricted to the language of the classical logic of general awareness. In particular, a variable can not be interpreted as a formula containing a variable or a quantifier.

[2] Defined essentially as in first-order logic.

Note that satisfaction of a formula without free variables does not depend on the syntactic valuation at all; thus we write $(M, s) \models \phi$ whenever $(M, s, \mathcal{V}) \models \phi$ for any $\mathcal{V}$, for such a formula.

As an example, consider the formula $\phi = \forall x A_i x$. Given a state s in model M, we have that $(M, s) \models \forall x A_i x$ iff $(M, s, \mathcal{V}) \models \forall x A_i x$ for some arbitrary $\mathcal{V}$ iff for every $\mathcal{V}' \sim_x \mathcal{V}$ we have that $(M, s, \mathcal{V}') \models A_i x$ iff for every $\psi \in \mathcal{L}_n^{K,X,A}(\Phi)$ we have that $\psi \in \mathcal{A}_i(s)$.

Example 3 (Example 2 continued) *(Adapted from Halpern and Rêgo (2006)). Now we can take the last sentence in Example 1 into account in our model of the situation. Let $M_3 \in \mathcal{M}_n(\Phi, \mathcal{S}_n^{\forall,K,X,A}(\Phi, \mathcal{X}))$ be as M_2, except that we let the investor be aware of the fact that there is something the broker is aware of but the investor is not: $\mathcal{A}_1(s) = \{p, \exists x(A_2 x \wedge \neg A_1 x)\}$.*

The formulae in Example 2 continue to hold in M_3 as well. The following two formulae (from Halpern and Rêgo (2006)) illustrate reasoning about unawareness. We have that:

- $(M_3, s) \models X_1(\exists x(A_2 x \wedge \neg A_1 x))$*: the investor explicitly knows that there is something that the broker is aware of but he is not*
- $(M_3, s) \models \neg X_2(\exists x(A_2 x \wedge \neg A_1 x))$*: the broker does not explicitly know that there is something he is aware of but the investor is not*

Let $\mathbf{K}_{n,\forall}$ be the axiom system over the language $\mathcal{L}_n^{\forall,K,X,A}(\Phi, \mathcal{X})$ consisting of the following axioms and rules:

Prop all propositional tautologies

K $K_i(\phi \rightarrow \psi) \rightarrow (K_i \phi \rightarrow K_i \psi)$

A0 $X_i \phi \leftrightarrow K_i \phi \wedge A_i \phi$

1$_\forall$ $\forall x \phi \rightarrow \phi[x/\psi]$ if ψ is quantifier free and substitutable[3] for x in ϕ

K$_\forall$ $\forall x(\phi \rightarrow \psi) \rightarrow (\forall x \phi \rightarrow \forall x \psi)$

N$_\forall$ $\phi \rightarrow \forall x \phi$ if x is not free in ϕ

Barcan $\forall x K_i \phi \rightarrow K_i \forall x \phi$

Gen$_\forall$ From ϕ infer $\forall x \phi$

MP From ϕ and $\phi \rightarrow \psi$ infer ψ

Gen From ϕ infer $K_i \phi$

Furthermore, given the following three extra axioms,

[3]Substitutable means that no free variable of ψ becomes bound as a result of the substitution.

T $K_i \phi \rightarrow \phi$

4 $K_i \phi \rightarrow K_i K_i \phi$

5 $\neg K_i \phi \rightarrow K_i \neg K_i \phi$

$\mathbf{K}_{n,\forall}^{\mathcal{C}}$ is the system obtained by adding axioms $\mathcal{C}$ to $\mathbf{K}_{n,\forall}$, where $\mathcal{C} \subseteq \{T, 4, 5\}$. It is well known that $T, 4$ and 5 correspond to the accessibility relations being reflexive, transitive and Euclidian, respectively.

Theorem 1 (Halpern and Rêgo (2006)) *Let $\mathcal{C} \subseteq \{T, 4, 5\}$ and let C be the corresponding subset of $\{r, t, e\}$. If Φ is countably infinite, $\mathbf{K}_{n,\forall}^{\mathcal{C}}$ is a sound and complete axiomatisation of the language $\mathcal{L}_n^{\forall,K,X,A}(\Phi, \mathcal{X})$ with respect to $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{\forall,K,X,A}(\Phi, \mathcal{X}))$.*

3 A Logic of Full and Relative Awareness

The logic of full and relative awareness is, like the HR logic, an extension of the logic of general awareness. Unlike the HR logic, it does not have variables or explicit quantification.

The language $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ is defined by the following grammar:

$$\phi ::= p \mid \phi_1 \wedge \phi_2 \mid \neg \phi \mid K_i \phi \mid X_i \phi \mid A_i \phi \mid C_i \mid R_{ij}$$

where $p \in \Phi$ and $i, j \in [1, n]$. Note that the two new connectives C_i and R_{ij} are nullary (they don't take any arguments). C_i is intended to mean that agent i has full awareness. R_{ij} is intended to mean that agent j has relative awareness with respect to agent i, i.e., that j is aware of everything i is aware of.

Satisfaction of $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ formulae is defined in relation to an awareness structure $M \in \mathcal{M}_n(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ and a state s of M. The following two clauses describe the new constructs, the clauses for the rest of the language are as before.

$$\begin{array}{lll} (M, s) \models C_i & \Leftrightarrow & \mathcal{A}_i(s) = \mathcal{L}_n^{C,R,K,X,A}(\Phi) \\ (M, s) \models R_{ij} & \Leftrightarrow & \mathcal{A}_i(s) \subseteq \mathcal{A}_j(s) \end{array}$$

Since $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ is infinite, C_i cannot be expressed by a finite conjunction of the form $A_i \phi_1 \wedge A_i \phi_2 \wedge \cdots$. $\neg C_i$ means that there exists a formula $\phi \in \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ such that $\phi \notin \mathcal{A}_i(s)$. Thus, $X_i \neg C_i$ expresses knowledge of unawareness: agent i explicitly knows that there is something he is unaware of. R_{ij} means that i's awareness set is included in j's awareness set, that j is aware of everything i is aware of. $\neg R_{ij}$ means that there is something i is aware of but j is not.

It is possible that $K_i \neg C_i$ is true, without there being any ϕ such that $K_i \neg A_i \phi$ is true, and it is possible that $K_i \neg R_{ji}$ is true without there being any ϕ such that $K_i(A_j\phi \wedge \neg A_i\phi)$ is true.

Example 4 (Example 3 continued) *Like in Example 3, we extend the awareness structures of Example 2 to take awareness about unawareness into account in order to model the fact described in the last sentence in Example 1*[4]*. The fact that there is something that the broker is aware of but the investor is not aware of can now be expressed by the formula* $\neg R_{21}$*. Thus, we let* $M_4 \in \mathcal{M}_n(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ *be like* M_2*, except that we set* $\mathcal{A}_1(s) = \{p, \neg R_{21}\}$*.*

The two formulae can now be expressed as follows. We have that:

- $(M_4, s) \models X_1(\neg R_{21})$
- $(M_4, s) \models \neg X_2(\neg R_{21})$

Note that the logic is not compact. As a counter example take the theory $\{\neg C_i\} \cup \{A_i\phi : \phi \in \mathcal{L}_n^{C,R,K,X,A}(\Phi)\}$, or the theory $\{\neg R_{ij}\} \cup \{\neg A_i\phi \vee A_j\phi : \phi \in \mathcal{L}_n^{C,R,K,X,A}(\Phi)\}$.

In the next section, we present an axiomatisation of the logic.

4 Axiomatisation

Let $\mathcal{S}$ be the axiom system consisting of the following axioms and inference rules, over the language $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$:

Prop all propositional tautologies

K $K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$

A0 $X_i\phi \leftrightarrow K_i\phi \wedge A_i\phi$

A1 $R_{ij} \to (A_i\phi \to A_j\phi)$

A2 R_{ii}

A3 $R_{ij} \wedge R_{jk} \to R_{ik}$

C1 $C_i \to A_i\phi$

C2 $C_i \to R_{ji}$

C3 $(C_i \wedge R_{ij}) \to C_j$

MP From ϕ and $\phi \to \psi$ infer ψ

Gen From ϕ infer $K_i\phi$

Prop-A0, MP and Gen axiomatise the logic of general awareness (Fagin and Halpern, 1988). A1 says that relative awareness implies that the other agents' awareness of a particular formula again implies awareness of the same formula. A2 and A3 say that relative awareness is reflexive and transitive, respectively. C1 says that full awareness implies awareness of any particular formula. C2 says that full awareness implies relative awareness (wrt. any other agent), and C3 says that relative awareness implies full awareness in the case that the other agent has full awareness.

Furthermore, $\mathcal{S}^{\mathcal{C}}$ is the system obtained by adding axioms $\mathcal{C}$ to $\mathcal{S}$, where $\mathcal{C} \subseteq \{T, 4, 5\}$.

Theorem 2 (Soundness and Weak Completeness) *Let* $\mathcal{C} \subseteq \{T, 4, 5\}$ *and let* C *be the corresponding subset of* $\{r, t, e\}$*.* $\mathcal{S}^{\mathcal{C}}$ *is a sound and weakly complete axiomatisation of the language* $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ *with respect to* $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$*.*

Proof. Soundness is straightforward.

For completeness, let ϕ be a $\mathcal{S}^{\mathcal{C}}$ consistent formula. We will show that ϕ is satisfiable in $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$, which completes the proof.

Build a canonical (standard) Kripke structure $M^c = (S^c, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ in the standard way:

- S^c is the set of all maximal $\mathcal{S}^{\mathcal{C}}$ consistent sets of formulae
- $(s,t) \in \mathcal{K}_i$ iff $K_i\psi \in s$ implies that $\psi \in t$, for all formulae ψ
- $p \in \pi(s)$ iff $p \in s$

To get a proper model $M = (S^c, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n)$, we extend M^c with awareness sets for each state. Let $s \in S^c$ and i be an agent. $\mathcal{A}_i(s)$ is defined as follows. Let $\mathit{Subf}(\phi)$ denote the set of subformulae of ϕ (including ϕ itself). Choose some $\psi_1, \ldots, \psi_n \in \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ such that $\psi_j \notin \mathit{Subf}(\phi)$ for any $j \in [1,n]$. We will refer to $\psi_1, \ldots, \psi_n$ as witness formulas. We now proceed in n steps; in each step i we define an awareness set X_k^i for each agent k:

- $X_i^0 = \begin{cases} \mathcal{L}_n^{C,R,K,X,A}(\Phi) & C_i \in s \\ \{\psi : A_i\psi \in s\} \cap \mathit{Subf}(\phi) & \text{otherwise} \end{cases}$
- $X_k^i = \begin{cases} X_k^{i-1} \cup \{\psi_i\} & R_{ik} \in s \text{ and there is a } j \text{ such that } \neg R_{ij} \in s \\ X_k^{i-1} & \text{otherwise} \end{cases}$

[4]Note that it does not make sense to use the awareness structure M_3 of Example 3 directly to interpret $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$, because this structure models awareness of formulae involving variables and quantifiers.

Set $\mathcal{A}_i(s) = X_i^n$.

If $r \in C$, $T \in \mathcal{C}$ ensures that each $\mathcal{K}_i$ is reflexive, and similarly for $t/4$ and $e/5$ (can be shown in the standard way). Thus, $M \in \mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$. We now show a truth lemma: for every formula $\psi \in \mathit{Subf}(\phi)$,

$$(M, s) \models \psi \Leftrightarrow \psi \in s$$

and then we are done, because $\phi \in \mathit{Subf}(\phi)$ and $\phi \in s_\phi$ for some $s_\phi \in S^c$, and thus $(M, s_\phi) \models \phi$.

The proof is by structural induction over $\psi \in \mathit{Subf}(\phi)$:

- $\psi = A_i\gamma$: for the direction to the left, $A_i\gamma \in s$ implies that $\gamma \in X_i^0 \subseteq \mathcal{A}_i(s)$. For the direction to the right, let $\gamma \in X_i^n$. If $C_i \in s$ then $A_i\gamma \in s$ by C1. If $C_i \notin s$, it must be the case that $\gamma \in X_i^0$ since $\gamma \in \mathit{Subf}(\phi)$ (γ cannot be one of the witness formulas ψ_j), and thus $A_i\gamma \in s$.

- $\psi = R_{ij}$: for the direction to the left, let $R_{ij} \in s$. In the case that $C_i \in s$, $C_j \in s$ by C3, and thus $\mathcal{A}_i(s) = \mathcal{A}_j(s) = \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ which implies that $(M, s) \models \psi$. Assume that $C_i \notin s$. Let $\gamma \in \mathcal{A}_i(s)$. We must show that $\gamma \in \mathcal{A}_j(s)$. Either $\gamma \in X_i^0$, or γ was added in step l for some l (i.e., $\gamma = \psi_l$). In the first case $A_i\gamma \in s$, so $A_j\gamma \in s$ by A1, and $\gamma \in X_j^0 \subseteq \mathcal{A}_j(s)$ since $\gamma \in \mathit{Subf}(\phi)$. In the second case $R_{li} \in s$, and there exists some m such that $\neg R_{lm} \in s$. Since $R_{li} \in s$ and $R_{ij} \in s$, $R_{lj} \in s$ by A3. That means that $\gamma = \psi_l$ was included in X_j^l as well in step l (the condition that there must be an m such that $\neg R_{lm} \in s$ holds). Thus, $\gamma \in \mathcal{A}_j(s)$.

  For the direction to the right, let $R_{ij} \notin s$, we must show that $\mathcal{A}_i(s) \not\subseteq \mathcal{A}_j(s)$. $\neg R_{ij} \in s$, and since $R_{ii} \in s$ by A2, $\psi_i \in X_i^i$ by construction. $\psi_i \notin \mathit{Subf}(\phi)$, and by C2 $\neg C_j \in s$, so if it were the case that $\psi_i \in \mathcal{A}_j(s)$ the only possibility is that $\psi_i \in X_j^i$. But then it would have to be the case that $R_{ij} \in s$, which is not true. Thus, $\psi_i \notin \mathcal{A}_j(s)$, and since $\psi_i \in \mathcal{A}_i(s)$ we have that $\mathcal{A}_i(s) \not\subseteq \mathcal{A}_j(s)$.

- $\psi = C_i$: For the direction to the left, let $C_i \in s$. $\mathcal{A}_i(s) = \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ by construction, so $(M, s) \models C_i$.

  For the direction to the right, let $\mathcal{A}_i(s) = \mathcal{L}_n^{C,R,K,X,A}(\Phi)$. The only way that can happen is when $C_i \in s$ (otherwise, $\mathcal{A}_i(s) \subseteq \mathit{Subf}(\phi) \cup \{\psi_1, \ldots, \psi_n\}$).

- $\psi = K_i\gamma$: this case can be shown in the standard way.

  Let $K_i\gamma \in s$. To show that $(M, s) \models K_i\gamma$, consider an arbitrary t such that $\mathcal{K}_i(s, t)$. By the definition of $\mathcal{K}_i$, $\gamma \in t$, and by the inductive hypothesis ($\gamma \in \mathit{Subf}(\phi)$) $(M, t) \models \gamma$. Hence, $(M, s) \models K_i\gamma$.

  Let $K_i\gamma \notin s$. We will find a t with $\mathcal{K}_i(s, t)$ such that $(M, t) \not\models \gamma$. This will show that $(M, s) \not\models K_i\gamma$. Consider the set $\{\neg\gamma\} \cup \{\chi : K_i\chi \in s\}$. This set is consistent (otherwise $\vdash_{\mathcal{S}} \chi_1 \wedge \ldots \wedge \chi_k \rightarrow \gamma$ for some $\chi_1, \ldots, \chi_k$ from this set, hence $\vdash_{\mathcal{S}} K_i\chi_1 \wedge \ldots \wedge K_i\chi_k \rightarrow K_i\gamma$, which would force $K_i\gamma \in s$). So, it can be extended to a mcs t. Since $\neg\gamma \in t$, $\gamma \notin t$, and by the inductive hypothesis $(M, t) \not\models \gamma$.

- The cases for atomic propositions, $\neg$ and $\wedge$ are straightforward.

□

5 Relationship to the HR Logic

Here we briefly comment on the relative meaning of $\forall x A_i x$ and $\forall x(A_i x \rightarrow A_j x)$ in the HR logic, and full and relative awareness formulas C_i and R_{ij} in our logic. One might hope to define an embedding of our logic into the HR logic by using the straightforward translation of C_i as $\forall x A_i x$ and R_{ij} as $\forall x(A_i x \rightarrow A_j x)$. However, this would not work for the following reason. $\forall x A_i x$ means, in the HR logic, that i is aware of ϕ for every $\phi \in \mathcal{L}_n^{K,X,A}(\Phi)$ in the classical logic of general awareness. There might still exist sentences which i is *not* aware of, such as $\forall x A_i x$ itself (which is not in $\mathcal{L}_n^{K,X,A}(\Phi)$). Conversely, $\forall x \neg A_i x$ does not mean that i's awareness set is empty, it strictly speaking means that i's awareness set does not contain formulae from $\mathcal{L}_n^{K,X,A}(\Phi)$. i might still be aware of other formulae. This means that a formula such as $(\forall x A_i x) \wedge (\neg A_i \forall x A_i x)$ is consistent in the HR logic. In our logic, however, $C_i \wedge \neg A_i C_i$ is *not* consistent. This point is illustrated by the C1 and the $1_\forall$ axioms of the respective logics. The former is unrestricted; full awareness means awareness of *any* formula, and, e.g., $C_i \rightarrow A_i C_i$ is an instance. The latter is restricted; for example is $\forall x A_i x \rightarrow A_i \forall x A_i x$ not an instance. Thus, C_i does not correspond to $\forall x A_i x$ in the HR logic (and similarly for relative awareness).

Thus, while it is obvious that the HR logic is more expressive than our logic, it is also true that our logic can express properties of (un)awareness which cannot be expressed in the HR logic. An example is "agent i is not aware of anything".

6 Decidability and Complexity

We are going to show that the satisfiability problem for $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ for any $C \subseteq \{r, t, e\}$

(or, equivalently, the validity problem with respect to $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$, or the derivability problem with respect to $\mathcal{S}^{\mathcal{C}}$ where $\mathcal{C} \subseteq \{T, 4, 5\}$), is decidable. We do this by a slight adaptation of the corresponding proof for multi-modal logics of knowledge (see, e.g., Halpern and Moses (1992)). Namely, we show that each consistent formula ϕ has a satisfying model where the number of states is bounded by the size of the formula, and the size of each agent's awareness set is either equal to the whole language, or is also bounded by the size of the formula.

Definition 1 (ϕ-bounded size) *With every formula $\phi \in \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ we associate an arbitrary but fixed set $\{\psi_1, \ldots, \psi_n\} \subset \mathcal{L}_n^{C,R,K,X,A}(\Phi)$ such that $\psi_j \notin Subf(\phi)$ for $1 \leq j \leq n$ where n is the number of agents.*

*Let ϕ be a formula of $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$. A model $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ has ϕ-*bounded size *iff*

- *For any $s \in S$ and any agent i, $\mathcal{A}_i(s)$ is either equal to $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ or $\mathcal{A}_i(s) \subseteq Subf(\phi) \cup \{\psi_1, \ldots, \psi_n\}$. Note that in the latter case $\mathcal{A}_i(s)$ is finite.*
- *$|S| \leq f(n, |\phi|)$ for some effectively computable function f.*

First, observe that each $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ satisfiable formula ϕ is satisfiable in a ϕ-bounded size model:

Lemma 1 *Let $C \subseteq \{r, t, e\}$.*
Every $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$-satisfiable formula ϕ has a ϕ-bounded size model $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ such that

- *for some s in S, $(M, s) \models \phi$*
- *each $\mathcal{K}_i$ satisfies the conditions C*

Proof. Given a satisfiable formula ϕ, we construct a model M' in exactly the same way as in the proof of Theorem 2. The awareness sets in this model satisfy the first property of ϕ-bounded size models. To obtain the bound on the number of states, we apply a filtration technique to M'. We define an equivalence relation on the set of states S' of M' as follows: $s \equiv t$ if s and t agree on the truth values of formulas from $Subf(\phi)$ and for every i, $\mathcal{A}_i(s) = \mathcal{A}_i(t)$. We will denote the result of filtration of M' by $M = (S, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n)$. The set of states S of M is the set of equivalence classes $[s]$ with respect to $\equiv$ from M'. The assignment π is M is defined in the standard way, namely $p \in \pi([s])$ if $p \in Subf(\phi)$ and $(M', s) \models p$. We do not change the awareness sets: $\mathcal{A}_i([s]) = \mathcal{A}_i(s)$, for every i. Finally, the accessibility relations $\mathcal{K}_i$ in M, for every i, are defined as usual in a filtration corresponding to a set of conditions $C \subseteq \{r, t, e\}$, so that they satisfy the following conditions (see Blackburn et al. (2001)):

F1 If $\mathcal{K}'_i(s, t)$ then $\mathcal{K}_i([s], [t])$

F2 if $\mathcal{K}_i([s], [t])$ then for all $K_i\psi \in Subf(\phi)$, if $(M, s) \models K_i\psi$ then $(M, t) \models \psi$.

All $C \subseteq \{r, t, e\}$ admit filtration (that is, a suitable definition for $\mathcal{K}_i$ exists and $\mathcal{K}_i$ in the resulting model satisfies C). The reader is referred to e.g. Blackburn et al. (2001) for possible definitions. For example, filtration for the equivalence relation satisfying $\{r, t, e\}$ is defined as follows: $\mathcal{K}_i([s], [t])$ iff for every $K_i\psi \in Subf(\phi)$, $(M', s) \models K_i\psi \Leftrightarrow (M', t) \models K_i\psi$.

Filtration of a model M' satisfying ϕ gives us a model $M \in \mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ for ϕ, where the size of S is bounded (admittedly, by a rather large number). Namely, observe that there are at most $2^{|\phi|}$ different states with respect to truth assignments to the subformulae of ϕ, and each of those states can have at most $2^{|\phi|+n+1}$ choices for each of the n awareness sets, which is $2^{n(|\phi|+n+1)}$ choices in total. The maximal number of possible states in M is therefore $2^{|\phi|} \times 2^{n(|\phi|+n+1)}$, or at most $2^{2n(|\phi|+n)}$, which is $2^{O(|\phi|)}$ if we treat the number of agents n as a constant.

The inductive proof that for every $\psi \in Subf(\phi)$, and every s, $(M', s) \models \psi \;\Leftrightarrow\; (M, [s]) \models \psi$ is standard.

$\psi = p$: trivial from the definition of π.

A_i, R_{ij}, C_i: trivial since $\mathcal{A}_i(s) = \mathcal{A}_i([s])$, for every i.

$\neg, \wedge$: by standard induction.

$\psi = K_i\gamma$.

Assume $(M', s) \models K_i\gamma$. Suppose $\mathcal{K}_i([s], [t])$. We need to show that $(M, [t]) \models \gamma$. By the second property of filtrations **F2**, $(M', s) \models K_i\gamma$ and $\mathcal{K}_i([s], [t])$ implies $(M', t) \models \gamma$. By the inductive hypothesis, this implies $(M, [t]) \models \gamma$.

Assume $(M, [s]) \models K_i\gamma$. Then for every $[t]$ such that $\mathcal{K}_i([s], [t])$, $(M, [t]) \models \gamma$. Let $\mathcal{K}'_i(s, t)$. We need to show that $(M', t) \models \gamma$. By the first property of filtrations **F1**, $\mathcal{K}'_i(s, t)$ implies $\mathcal{K}_i([s], [t])$. We know that $(M, [t]) \models \gamma$, so by the inductive hypothesis, $(M', t) \models \gamma$, which is what we needed to show.

□

Decidability follows immediately.

Theorem 3 (Decidability) *The satisfiability problem for $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$, where $C \subseteq \{r, t, e\}$, is decidable.*

Proof. Decidability is entailed by the following facts.

- Each ϕ-bounded size model is finitely representable. Although an awareness set may be equal to the complete language $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$, this fact can be finitely represented by using some symbol representing $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$.
- For a given formula ϕ, there is a fixed finite number of different ϕ-bounded size models.
- Given a formula ϕ and a finite description of a ϕ-bounded size model, we can effectively check whether it is indeed a representation of a $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ model and whether it satisfies ϕ.

This gives a decision procedure for the satisfiability problem for $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ as follows. Given a formula ϕ, let k be the number of ϕ-bounded size models. Enumerate these models, and for each check whether it satisfies ϕ or not. This procedure will terminate after we have checked ϕ against each of the k models. By Lemma 1, ϕ is satisfiable if and only if it was satisfied by one of the ϕ-bounded size models. □

Decidability via bounded model property gives us a rather high upper bound on the complexity of the satisfiability problem of our logic. We can however show that this problem is no harder than the corresponding problem for classical multi-modal logics of knowledge:

Theorem 4 (Complexity) *The satisfiability problem for $\mathcal{M}_n^C(\Phi, \mathcal{L}_n^{C,R,K,X,A}(\Phi))$ for any $C \subseteq \{r, t, e\}$ is PSPACE-complete.*

Proof. (Sketch) PSPACE-hardness follows from the results for corresponding multi-modal logics, see Halpern and Moses (1992).

To show PSPACE upper bound, we adapt the tableau algorithm of Halpern and Moses (1992) for logics K_n^C, $C \subseteq \{T, 4, 5\}$. The idea is to show that we can extend the step for forming a fully expanded propositional tableau by expansion rules for formulas of the form $A_i\psi$, C_i and R_{ij} in such a way that information about every node in the tableau can still be stored using space polynomial in $|\phi|$ (the formula for which we are constructing a tableau) and the number of agents n. The modal depth of the tableau is not affected. Then we add additional conditions for when a node is marked as unsatisfiable. Finally, we show that for every formula ψ, $\psi \in L(s)$ implies $T, s \models \psi$ and $\neg\psi \in L(s)$ implies $T, s \models \neg\psi$, where T is the model corresponding to the tableau, s is a node marked as satisfiable, and $L(s)$ is its labelling.

The additional expansion rules are:

rel-awareness if $A_i\psi, R_{ij} \in L(s)$, then create a successor s' of s with $L(s') = L(s) \cup \{A_j\psi\}$

transitivity if $R_{ij}, R_{jk} \in L(s)$, then create a successor s' of s with $L(s') = L(s) \cup \{R_{ik}\}$

full-awareness if $R_{ij}, C_i \in L(s)$, then create a successor s' of s with $L(s') = L(s) \cup \{C_j\}$

Additional conditions for when a node is marked as unsatisfiable are:

mark s as unsatisfiable if $L(s)$ contains $\neg R_{ii}$ for any i

mark s as unsatisfiable if $L(s)$ contains C_i and $\neg A_i\psi$ for any i and ψ.

Note that to store the node information in the extended language it is not enough to have a bit vector of length $2|\phi|$ to represent which of ϕ's subformulae or their negations are present, but we also need $n|\phi|$ bits to represent extra formulas which may be added by step **rel-awareness**, $2n^2$ bits for the formulas of the form R_{ij} added by **transitivity** and $2n$ for the formulas of the form C_i which may be added by **full-awareness**. However, the resulting space usage is still polynomial in $|\phi|$ and n (or in $|\phi|$ if we are treating n as a constant).

Finally, we need to show that if a node s is marked as satisfiable, then we can construct awareness sets $\mathcal{A}_1(s), \ldots, \mathcal{A}_n(s)$ so that for all formulas $\psi \in Subf(\phi)$ of the form $A_i\gamma, C_i, R_{ij}$,

$\psi \in L(s)$ implies $T, s \models \psi$, and $\neg\psi \in L(s)$ implies $T, s \models \neg\psi$.

We construct $\mathcal{A}_i(s)$ essentially as in the proof of Theorem 2. We use witness formulas $\psi_1, \ldots, \psi_n$ which are not in $Subf(\phi)$, and we set $X_i^0(s)$ to be either $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ if $C_i \in L(s)$, or the set of $\{\gamma : A_i\gamma \in L(s)\}$ otherwise. At the step corresponding to agent i, if $R_{ik}, \neg R_{ij} \in L(s)$ then we add ψ_i to X_k^i, for every such k, and to X_i^i. We set $\mathcal{A}_i(s)$ to be X_i^n.

Now consider the three cases we have:

$\psi = A_i\gamma$. If $A_i\gamma \in L(s)$, then $\gamma \in \mathcal{A}_i(s)$,so $T, s \models A_i\gamma$. If $\neg A_i\gamma \in L(s)$, then $\mathcal{A}_i(s)$ is not equal to $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$ (because s is consistent, so $C_i \notin L(s)$) and $\mathcal{A}_i(s)$ does not contain γ (because again due to consistency $A_i\gamma \notin L(s)$). So $T, s \models \neg A_i\gamma$.

$\psi = C_i$. If $C_i \in L(s)$, then $\mathcal{A}_i(s) = \mathcal{L}_n^{C,R,K,X,A}(\Phi)$, so $T, s \models C_i$. If $\neg C_i \in L(s)$ then $C_i \notin L(s)$, so $\mathcal{A}_i(s) \neq \mathcal{L}_n^{C,R,K,X,A}(\Phi)$, so $T, s \models \neg C_i$.

$\psi = R_{ij}$. Suppose by contradiction that $R_{ij} \in L(s)$ and $\mathcal{A}_i(s) \not\subseteq \mathcal{A}_j(s)$. By construction and consistency of s, $\mathcal{A}_i(s)$ and $\mathcal{A}_j(s)$ are not equal to $\mathcal{L}_n^{C,R,K,X,A}(\Phi)$. So the formula which is in $\mathcal{A}_i(s)$ but not in $\mathcal{A}_j(s)$ is either some γ such that $A_i\gamma \in L(s)$, or one of the witness formulas. The first case is excluded by the **rel-awareness** rule which forces $A_j\gamma \in L(s)$, hence in $\mathcal{A}_j(s)$. The second case is excluded by the **transitivity** rule and the way we add witnesses.

Let $\neg R_{ij} \in L(s)$. Then we added a witness ψ_i to X_i^i, which is not in $\mathcal{A}_j(s)$. So $T, s \models \neg R_{ij}$.

□

7 Conclusions

We have pointed out that the full expressiveness of unrestricted quantification is not needed to express knowledge of unawareness in the motivating examples of Halpern and Rêgo (2006), that quantification restricted to *full* and *relative* awareness is sufficient, and that the logic of full and relative awareness is decidable (in PSPACE), and we have presented a sound and complete axiomatisation of that logic.

By negating full and relative awareness, we have seen that we can express the fact that *there is at least one* fact the agent is not aware of, and *there is at least one* fact the agent is aware of and the other agent is not aware of, respectively. This could possibly be generalised to *there is at least n*, for arbitrary natural numbers n. We studied such "at least n" operators in (Ågotnes and Alechina, 2006), where we investigated an epistemic language interpreted in purely syntactic structures (Fagin et al., 1995), extended with an operator $min(n)$ meaning that the agent explicitly knows at least n formulae. Also, it would be interesting to investigate variants of the logic by imposing restrictions on the awareness sets, such as awareness being *generated by primitive propositions* (Halpern, 2001).

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