What can we achieve by arbitrary announcements? A dynamic take on Fitch's knowability

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Abstract

Public announcement logic is an extension of multi-agent epistemic logic with dynamic operators to model the informational consequences of announcements to the entire group of agents. We propose an extension of public announcement logic with a dynamic modal operator that expresses what is true after *any* announcement: $\Box\varphi$ expresses that φ is true after an arbitrary announcement ψ. As this includes the trivial announcement $\top$, one might as well say that $\Box\varphi$ expresses what *remains* true after any announcement: it therefore corresponds to truth persistence after (definable) relativisation. The dual operation $\Diamond\varphi$ expresses that there is an announcement after which φ. This gives a perspective on Fitch's knowability issues: for which formulas φ does it hold that $\varphi \rightarrow \Diamond K\varphi$? We give various semantic results, and we show completeness for a Hilbert-style axiomatisation of this logic.

1 INTRODUCTION

One motivation to formalise the dynamics of knowledge is to characterise how truth or knowledge conditions can be realised by new information. From that perspective, it seems unfortunate that in public announcement logic [13, 6, 17] it may come to pass that a true formula becomes false because it is announced. The prime example is the new information expressed by the Moore-sentence 'atom p is true and you (agent a) do not know that', formalised by $p \wedge \neg K_a p$ [12, 8], but there are many other examples [16]. After the Moore-sentence is announced, you know that p is true, so $p \wedge \neg K_a p$ is now false. Worse, no additional announcement or sequence of announcements can make it true again. Also, the Moore-sentence cannot become known. But, for example, true facts p can always become known. The issues of what can become true and known are also known as reachability and knowability, respectively, and the 'Fitch-paradox' addresses the problematic question whether what is true can become known. For example, see van Benthem in [15] or, for further references, [3].

Consider an extension of public announcement logic wherein we can express what becomes true, whether known or not, without explicit reference to announcements realising that. Let us work our way upwards from a concrete announcement. When p is true, it becomes known by announcing it. Formally, in public announcement logic, $p \wedge [p]K_a p$. This is equivalent to

$$\langle p \rangle K_a p$$

which stands for 'the announcement of p can be made and after that the agent knows p'. More abstractly this means that there is a announcement ψ, namely $\psi = p$, that makes the agent know p, slightly more formal:

$$\text{there is a formula } \psi \text{ such that } \langle \psi \rangle K_a p$$

We introduce a dynamic modal operator that expresses that:

$$\Diamond K_a p$$

Obviously, the truth of this expression depends on the model: p has to be true. In case p is false, we can achieve $\Diamond K_a \neg p$ instead. The formula $\Diamond(K_a p \vee K_a \neg p)$ is valid.

We overlooked a 'detail' of the semantics. The condition '$\Diamond\varphi$ is true iff there is a ψ such that $\langle\psi\rangle\varphi$ is true' (for some state of the world in a Kripke model) is not well-defined. For example, the announced formula ψ may be the formula $\Diamond\varphi$ itself! We therefore need a syntactic restriction on announcements replacing a $\Diamond$ operator in a formula, or a semantic restriction enforced by the structures in which we interpret arbitrary announcements. We propose that such announcements may not contain $\Diamond$ operators. With that restriction, and also using that public announcement logic (without common knowledge) is equally expressive as multi-agent epistemic logic, the language is well-defined. The corresponding logic is called arbitrary public announcement logic, or in short, *arbitrary announcement logic*.

In Section 2 we define the logical language $\mathcal{L}_{apal}$ and its semantics. Section 3 shows various semantic results, including a 'knowable' fragment of the language and an expressivity result. In Section 4 we provide a Hilbert-style axiomatisation of arbitrary announcement logic.

2 SYNTAX AND SEMANTICS

Assume a finite set of agents A and a countably infinite set of atoms P.

Definition 1 (Language) The language $\mathcal{L}_{apal}$ of arbitrary public announcement logic is inductively defined as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

where $a \in A$ and $p \in P$. Additionally, $\mathcal{L}_{pal}$ is the language without inductive construct $\Box\varphi$, $\mathcal{L}_{el}$ the language without as well $[\varphi]\varphi$, and $\mathcal{L}_{pl}$ the language without as well $K_a\varphi$.

The languages $\mathcal{L}_{pal}$, $\mathcal{L}_{el}$, and $\mathcal{L}_{pl}$ are, of course, those of public announcement logic, epistemic logic, and propositional logic, respectively. For $K_a\varphi$, read 'agent a knows that φ'. For $[\varphi]\psi$, read 'after public announcement of φ, ψ is true'. For $\Box\psi$, read 'after every public announcement, ψ is true'. Other propositional and epistemic connectives are defined by usual abbreviations. The dual of K_a is $\widehat{K}_a$, the dual of $[\varphi]$ is $\langle\varphi\rangle$, and the dual of $\Box$ is $\Diamond$. For $\widehat{K}_a\varphi$, read 'agent a considers it possible that φ', for $\langle\varphi\rangle\psi$, read 'announcement φ can be made after which ψ (is true)' and for $\Diamond\psi$, read 'there is an announcement after which ψ.' Write P_φ for the set of atoms occurring in the formula φ. Given some $P' \subseteq P$, $\mathcal{L}_x(P')$ is the logical language $\mathcal{L}$ ($\mathcal{L}_{apal}, \mathcal{L}_{el}, \dots$) restricted to atoms in P'.

Definition 2 (Structures) An *epistemic model* $M = (S, \sim, V)$ consists of a *domain* S of (factual) *states* (or 'worlds'), *accessibility* $\sim : A \rightarrow \mathcal{P}(S \times S)$, where each $\sim(a)$ is an equivalence relation, and a *valuation* $V : P \rightarrow \mathcal{P}(S)$. For $s \in S$, (M, s) is an *epistemic state* (also known as a pointed Kripke model). An *epistemic frame* $\mathbf{S}$ is a pair $(S, \sim)$. For a model we also write $(\mathbf{S}, V)$ and for a pointed model also $(\mathbf{S}, V, s)$.

For $\sim(a)$ we write $\sim_a$, and for $V(p)$ we write V_p; accessibility $\sim$ can be seen as a set of equivalence relations $\sim_a$, and V as a set of valuations V_p. Given two states s, s' in the domain, $s \sim_a s'$ means that s is indistinguishable from s' for agent a on the basis of its knowledge. We adopt the standard rules for omission of parentheses in formulas, and we also delete them in representations of structures such as (M, s) whenever convenient and unambiguous.

Definition 3 (Semantics) Assume an epistemic model $M = (S, \sim, V)$. The interpretation of $\varphi \in \mathcal{L}_{apal}$ is defined by induction. Note the restriction to epistemic formulas in the clause for $\Box\varphi$.

$$\begin{array}{lll}
M, s \models p & \text{iff} & s \in V_p \\
M, s \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models K_a\varphi & \text{iff} & \forall t \in S : s \sim_a t \text{ implies } M, t \models \varphi \\
M, s \models [\varphi]\psi & \text{iff} & M, s \models \varphi \text{ implies } M|\varphi, s \models \psi \\
M, s \models \Box\varphi & \text{iff} & \forall \psi \in \mathcal{L}_{el} : M, s \models [\psi]\varphi
\end{array}$$

In clause $[\varphi]\psi$ for public announcement, epistemic model $M|\varphi = (S', \sim', V')$ is defined as

$$\begin{array}{lll}
S' & = & \{s' \in S \mid M, s' \models \varphi\} \\
\sim'_a & = & \sim_a \cap (S' \times S') \\
V'_p & = & V_p \cap S'
\end{array}$$

Formula φ is valid in model M, notation $M \models \varphi$, iff for all $s \in S$: $M, s \models \varphi$. Formula φ is valid, notation $\models \varphi$, iff for all M (given the parameters A and P): $M \models \varphi$.

The dynamic modal operator $[\varphi]$ is interpreted as an epistemic state transformer. Announcements are assumed to be truthful, and this is commonly known by all agents. Therefore, the model $M|\varphi$ is the model M restricted to all the states where φ is true, including access between states. Similarly, the dynamic model operator $\Box$ is interpreted as an epistemic state transformer. Note that in the definiendum of $\Box\varphi$ the announcements ψ in $[\psi]\varphi$ are restricted to purely epistemic formulas $\mathcal{L}_{el}$. This is motivated in depth, below. For the semantics of the dual operators, we have that $M, s \models \Diamond\psi$ iff there is a $\varphi \in \mathcal{L}_{el}$ such that $M, s \models \langle\varphi\rangle\psi$. And we have that $M, s \models \langle\varphi\rangle\psi$ iff $M, s \models \varphi$ and $M|\varphi, s \models \psi$. Write $[\![\varphi]\!]_M$ for the denotation of φ in M, i.e., $\{s \in S \mid (M, s) \models \varphi\}$.

The set of validities in our logic is called *APAL*—and we also use *APAL* more informally (e.g., without specifying parameter sets of agents and atoms) to refer to arbitrary public announcement logic. Similarly for *PL* (propositional logic), *EL* (epistemic logic, a.k.a. $S5_n$ where $|A| = n$), and *PAL* (public announcement logic).

Example 4 A valid formula of the logic is $\Diamond(K_a p \vee K_a \neg p)$. To prove this, let (M, s) be arbitrary. Either $M, s \models p$ or $M, s \models \neg p$. In the first case, $M, s \models \Diamond(K_a p \vee K_a \neg p)$ because $M, s \models \langle p \rangle (K_a p \vee K_a \neg p)$ – the latter is true because $M, s \models p$ and $M|p, s \models K_a p$; in the second case, we analogously derive $M, s \models \Diamond(K_a p \vee K_a \neg p)$ because $M, s \models \langle \neg p \rangle (K_a p \vee K_a \neg p)$.

This example also nicely illustrates the order in which arbitrary objects come to light. The meaning of $\models \Diamond\varphi$ is (i) '*for all* (M, s) *there is a* ψ *such that* $M, s \models \langle\psi\rangle\varphi$'. This is really different from (ii) '*there is a* ψ *such that for all* (M, s)*,* $M, s \models \langle\psi\rangle\varphi$', which might on first sight be appealing to the reader, when extrapolating from the *incorrect* reading 'there is a ψ such that $\models \langle\psi\rangle\varphi$' of $\models \Diamond\varphi$. But, for example, there is no formula ψ in the language such that $\langle\psi\rangle(K_a p \vee K_a \neg p)$ is valid: in other words, (i) may be true, even when (ii) is false.

We now compare the given semantics for $\Box\varphi$ to two infelicitous alternatives, thus hoping to motivate our choice. The three options are (infelicitous alternatives are *-ed):

$$\begin{array}{rcll} M, s \models \Box\varphi & \text{iff} & \forall\psi \in \mathcal{L}_{el} : M, s \models [\psi]\varphi & \text{(Def. 3)} \\ *M, s \models \Box\varphi & \text{iff} & \forall\psi \in \mathcal{L}_{apal} : M, s \models [\psi]\varphi & \text{(intuitive)} \\ *M, s \models \Box\varphi & \text{iff} & \forall S' \subseteq S \text{ containing } s : M|S', s \models \varphi & \text{(structural)} \end{array}$$

The 'intuitive' version for the semantics of $\Box\varphi$ more properly corresponds to its intended meaning 'φ is true after arbitrary announcements'. This version is not well-defined, as $\Box\varphi$ is itself one such announcement.

The 'structural' version for the semantics of $\Box\varphi$ is more in accordance with one of Fine's proposals for quantification over propositional variables in modal logic [4]; his work strongly inspired our approach. This structural version is undesirable for our purposes as it does not preserve bisimilarity of structures: two bisimilar states can now be separated because they will be in different subdomains. In dynamic epistemic logics it is considered preferable that action execution preserves bisimilarity; this is because bisimilarity implies logical equivalence, and we tend of think of such actions as changing the theories describing those structures, just as in belief revision. For an example, consider the following structure M:

$$\begin{array}{ccc} \mathbf{1} & \text{——}a\text{——} & \mathbf{0} \\ |b & & |b \\ \underline{1} & \text{——}a\text{——} & 0 \end{array}$$

We have that $M, 1 \models \Diamond(K_a p \wedge \neg K_b K_a p)$ for the structural $\Box$-semantics, as $M|\{1, \mathbf{1}, \mathbf{0}\}, 1 \models K_a p \wedge \neg K_b K_a p$. On the other hand, for the $\Box$-semantics as defined, $M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, which can be easily seen as that formula is also false in $\underline{1}$——a——0, that is bisimilar to $(M, 1)$.

Concerning the semantics as defined, note that public announcement logic is equally expressive as multiagent epistemic logic [13], so our restriction corresponds to '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{pal}$.' Given that truth is relative to a model, the semantics for $\Box$ amounts to '$\Box\varphi$ is true in (M, s) iff φ is true in all epistemistically definable submodels of M.'

3 SEMANTIC RESULTS

3.1 VALIDITIES

Proposition 5 Let, $\varphi, \psi \in \mathcal{L}_{apal}$ be arbitrary. Then

1. $\models \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
2. $\models \Box\varphi \rightarrow \varphi$
3. $\models \Box\varphi \rightarrow \Box\Box\varphi$
4. $\models \varphi$ implies $\models \Box\varphi$
5. $\models K_a\Box\varphi \rightarrow \Box K_a\varphi$ (but *not* in the other direction)
6. $\models \Box p \leftrightarrow p$ (or, in general, for booleans)

Proof Here (and elsewhere) 'arbitrary' formulas instantiating $\Box$ or $\Diamond$ are always *implicitly* assumed to be 'of the right type', i.e.: $\mathcal{L}_{el}$ formulas.

1. Obvious.
2. Assume $M, s \models \Box\varphi$. Then in particular, $M, s \models [\top]\varphi$, and therefore (as $M, s \models \top$) $M, s \models \varphi$.
3. Let M and $s \in M$ be arbitrary. Assume $M, s \models \Diamond\Diamond\neg\varphi$. Then there are χ and χ' such that $M, s \models \langle\chi\rangle\langle\chi'\rangle\neg\varphi$. Using the validity (for arbitrary formulas) $[\varphi][\varphi']\varphi'' \leftrightarrow [\varphi \wedge [\varphi]\varphi']\varphi''$, we therefore have $M, s \models \langle\chi \wedge [\chi]\chi'\rangle\neg\varphi$, from which follows $M, s \models \Diamond\neg\varphi$.
4. Let M, s be arbitrary. We have to show that for arbitrary ψ (i.e. arbitrary $\psi \in \mathcal{L}_{pal}$): $M, s \models [\psi]\varphi$. From the assumption $\models \varphi$ follows $\models [\psi]\varphi$ by necessitation for $[\psi]$. Therefore also $M, s \models [\psi]\varphi$.
5. Let (M, s), φ, and $t \in M|\varphi$ with $t \sim_a s$ be arbitrary. We have to prove that $M|\psi, t \models \varphi$. Because state t is also in M, from the assumption $M, s \models K_a\Box\varphi$ and $s \sim_a t$ also in M follows $M, t \models \Box\varphi$. As ψ is true in t, $M|\psi, t \models \varphi$.

$\Box$

Also valid are $\models \Box\Diamond\varphi \to \Diamond\Box\varphi$ (McKinsey — MK) and $\models \Diamond\Box\varphi \to \Box\Diamond\varphi$ (Church-Rosser — CR). MK ($\Box\Diamond\varphi \to \Diamond\Box\varphi$) in conjunction with 4 ($\Box\varphi \to \Box\Box\varphi$) correspond to the frame property of *atomicity* (see for a reference [2, p.167,Ex.3.57]), defined as $\forall x \exists y(Rxy \wedge \forall z(Ryz \to z = y))$. In our terms, atomicity describes that one can always make a *most* informative announcement. CR corresponds to the frame property of 'confluence'. In our terms, this can be formulated as follows. Given two distinct (and true) announcements φ, ψ in some epistemic state (M, s), then there are subsequent announcements φ', ψ' such that $(M|\varphi|\varphi', s)$ is bisimilar to $(M|\psi|\psi', s)$.

Proposition 6 (MK is valid) $\models \Box\Diamond\varphi \to \Diamond\Box\varphi$

Proof We sketch the proof. Let M, s be arbitrary, and assume $M, s \models \Box\Diamond\varphi$. Consider the characteristic formula δ_s^φ of the valuation in s restricted to the (finite number of!) atoms occurring in φ. One can show (a technical report with details is in preparation) that after announcing this formula, arbitrary $\psi \in \mathcal{L}_{apal}(P_\varphi)$ (formulas only using atoms occurring in φ) will not change their value any more, i.e. $M|\delta_s^\varphi, s \models \psi \to \Box\psi$. This is because when we replace all atoms in ψ by their value in s, the resulting variable free proposition $\psi^\emptyset$ is clearly equivalent to ψ in the model and obviously does not change value after model restriction (note that $\Box\top \leftrightarrow \top$ and $\Box\bot \leftrightarrow \bot$).

From $M, s \models \Box\Diamond\varphi$ and $M, s \models \delta_s^\varphi$ follows $M|\delta_s^\varphi, s \models \Diamond\varphi$. From that and $M|\delta_s^\varphi, s \models \psi \to \Box\psi$ for arbitrary $\psi \in \mathcal{L}_{apal}$ then follows in two steps that $M|\delta_s^\varphi, s \models \Box\varphi$. Therefore $M, s \models \langle\delta_s^\varphi\rangle\Box\varphi$, thus $M, s \models \Diamond\Box\varphi$. $\Box$

Proposition 7 (CR is valid) $\models \Diamond\Box\varphi \to \Box\Diamond\varphi$

Proof A sketch of the proof is as follows. One can show (details omitted) that the effect of an announcement in a model can always be simulated by announcing a fresh atom, in the following sense: if $M, s \models \Diamond\varphi$, then there is an atom p not occurring in φ such that $M', s \models \langle p\rangle\varphi$ where M' *only* differs from M in the valuation of atom p.

With this result, CR follows easily: suppose that CR fails, i.e. there exist M, s and φ such that $M, s \models \Diamond\Box\varphi \wedge \Diamond\Box\neg\varphi$. Then there are $p, q \notin P_\varphi$ and a model M' that is like M except for the valuation of p and q, such that $M', s \models \langle p\rangle\Box\varphi \wedge \langle q\rangle\Box\neg\varphi$. We therefore also have $M', s \models \langle p\rangle[q]\varphi \wedge \langle q\rangle[p]\neg\varphi$ from which follows $M', s \models \langle p \wedge q\rangle(\varphi \wedge \neg\varphi)$, which is a contradiction. $\Box$

3.2 EXPRESSIVITY

If there is a single agent only, arbitrary announcement logic reduces to epistemic logic. But for more than one agent, it is strictly more expressive than public announcement logic (which—as we do not have common knowledge—is equally expressive as epistemic logic).

The following Proposition 8 will be helpful to show that in the single-agent case every formula is equivalent to an epistemic $\mathcal{L}_{el}$-formula. Note that it holds for an *arbitrary*, not necessarily singleton, set of agents A.

Proposition 8 Let $\varphi, \varphi_0, \ldots, \varphi_n \in \mathcal{L}_{pl}$ and $\psi \in \mathcal{L}_{apal}$.

1. $\models \Box\varphi \leftrightarrow \varphi$
2. $\models \Box\hat{K}_a\varphi \leftrightarrow \varphi$
3. $\models \Box K_a\varphi \leftrightarrow K_a\varphi$
4. $\models \Box(\varphi \vee \psi) \leftrightarrow (\varphi \vee \Box\psi)$
5. $\models \Box(\hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n) \leftrightarrow (\varphi_0 \vee K_a(\varphi_0 \vee \varphi_1) \vee \ldots \vee K_a(\varphi_0 \vee \varphi_n))$

Proof In the proof, we use the dual (diamond) versions of all propositions.

1. $\models \Diamond\varphi \leftrightarrow \varphi$
   This is valid because $\langle\psi\rangle\varphi \leftrightarrow \varphi$ is valid in *PAL*, for any ψ and boolean φ.

2. $\models \Diamond K_a\varphi \leftrightarrow \varphi$
   Right-to-left holds because $\varphi \to \langle\varphi\rangle K_a\varphi$ is valid in *PAL* for booleans. The other way round, $\models \Diamond K_a\varphi \to \varphi$ because $\Diamond K_a\varphi \to \Diamond\varphi$ is valid in *PAL*, and $\Diamond\varphi \leftrightarrow \varphi$ is valid in *PAL* as we have seen above (φ being boolean).

3. $\models \Diamond\hat{K}_a\varphi \leftrightarrow \hat{K}_a\varphi$
   Right-to-left holds because axiom T applies to $\Diamond$. Left-to-right holds because $\langle\psi\rangle\hat{K}_a\varphi \to \hat{K}_a\varphi$ is valid in *PAL* for booleans φ.

4. $\models \Diamond(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \Diamond\psi$
   From left-to-right: first, $\Diamond$ distributes over $\wedge$, and second, $\models \Diamond\varphi \leftrightarrow \varphi$ as we have established above. From right-to-left: $\varphi \wedge \Diamond\psi$ is equivalent to (apply case 1) $\Box\varphi \wedge \Diamond\psi$. Let χ be an announcement realising ψ in a given pointed structure (M, s), i.e. $M, s \models \langle\chi\rangle\psi$. This implies also that $M, s \models \chi$, and from that and $M, s \models \Box\varphi$ then follows $M, s \models \chi \wedge [\chi]\varphi$, i.e. $M, s \models \langle\chi\rangle\varphi$. Therefore $M, s \models \langle\chi\rangle(\varphi \wedge \psi)$ so also $M, s \models \Diamond(\varphi \wedge \psi)$.

5. $\models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1 \wedge \ldots \wedge \hat{K}_a\varphi_n) \leftrightarrow \varphi_0 \wedge \hat{K}_a\varphi_1 \wedge \ldots \wedge \hat{K}_a\varphi_n$
   We show this case for $n = 1$.

   $\Rightarrow$ Directly in the semantics. Let M, s be arbitrary and suppose $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$.

Let ψ be the epistemic formula such that $M, s \models \langle\psi\rangle(K_a\varphi_0 \wedge M_a\varphi_1)$. In the model $M|\psi$ we now have that $M|\psi, s \models K_a\varphi_0$ so $M|\psi, s \models \varphi_0$. Also $M|\psi, s \models \hat{K}_a\varphi_1$. Let t be such that $s \sim_a t$ and $M|\psi, t \models \varphi_1$. As $M|\psi, s \models K_a\varphi_0$, and $s \sim_a t$, also $M|\psi, t \models \varphi_0$. Therefore $M|\psi, t \models \varphi_0 \wedge \varphi_1$, and therefore $M|\psi, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$. So $M|\psi, s \models \varphi_0 \wedge M_a(\varphi_0 \wedge \varphi_1)$ and as φ_0 and φ_1 are booleans also $M, s \models \varphi_0 \wedge M(\varphi_0 \wedge \varphi_1)$.[1]

$\Leftarrow$ For the other direction, suppose $M, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$. Consider the model $M|\varphi_0$. Because $M, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$, and φ_1 is boolean, there must be a $tinM|\varphi_0$ such that $M|\varphi_0, t \models \varphi_1$. So $M|\varphi_0, s \models \hat{K}_a\varphi_1$. Also $M|\varphi_0, s \models K_a\varphi_0$, because φ_0 is boolean. So $M|\varphi_0, s \models K_a\varphi_0 \wedge \hat{K}_a\varphi_1$ and therefore $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$.

$\Box$

Now, we restrict ourselves to a single agent: let $A = \{a\}$. A formula is in *normal form* when it is a conjunction of disjunctions of the form $\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \ldots \vee K_a\varphi_n$. Every formula in single-agent epistemic logic (*K45* and therefore also in) *S5* is equivalent to a formula in normal form [10].

Proposition 9 Single agent arbitrary announcement logic is equally expressive as epistemic logic.

Proof We prove by induction on the number of occurrences of $\Box$, that every formula in arbitrary announcement logic is equivalent to a formula in epistemic logic. Put the epistemic formula in the scope of an innermost $\Box$ in normal form.[2] First, we distribute $\Box$ over the conjunction (proposition 5.1). We now get formulae of the form $\Box(\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n)$. These are reduced by application of a number of omitted technical results to formulas of form $\varphi_0 \vee K_a(\varphi_0 \vee \varphi_1) \vee \cdots \vee K_a(\varphi_0 \vee \varphi_n)$. $\Box$

Proposition 10 Arbitrary announcement logic is strictly more expressive than public announcement logic.

Proof The proof follows an abstract argument. Suppose the logics are equally expressive, in other words, that there is some reduction rule for arbitrary announcement such that $\Box\varphi$ can be reduced to an expression without $\Box$. Given the reduction of *PAL* to *EL*, this entails that every arbitrary announcement formula should be equivalent to an epistemic logical formula. Now the crucial observation is that this epistemic formula only contains a *finite* number of atomic propositions. We then construct models that cannot be distinguished in the restricted language, but can be distinguished in a language with more atoms.

So it remains to give a specific formula and a specific pair of models. Note that the formula must involve more than one agent, as single-agent arbitrary announcement logic is reducible to epistemic logic (see Proposition 9).

Consider the formula $\Diamond(K_ap \wedge \neg K_bK_ap)$. Assume, towards a contradiction, that it is equivalent to an epistemic logical formula ψ. W.l.o.g. we may assume that ψ only contains the atom p.[3] We now construct two different epistemic states (M, s) and (M', s') involving a *new* atom q such that $\Diamond(K_ap \wedge \neg K_bp)$ is true in the first but false in the second. We also take care that the two models are bisimilar with respect to the language without q. Therefore, the supposed 'reduction' is either true in both models or false in both models. Contradiction. Therefore, no such reduction exists.

The required models are as follows. Epistemic state $(M, 1)$ consists of the well-known model M where a cannot distinguish between states where p is true and false, but b can (but knows that a cannot, etc.), i.e., domain $\{0, 1\}$ with universal access for a and identity access for b, where p is only true at 1, and 1 is the actual state. Visualised as:

$$\underline{1} \text{———} a \text{———} 0$$

Epistemic state $(M', 10)$ consists of two copies of M, namely one where a new fact q is true and another one where q is false. In the actual state 10, q is false. We visualise this as follows

$$\begin{array}{ccc} 11 & \text{———}a\text{———} & 01 \\ |b & & |b \\ \underline{10} & \text{———}a\text{———} & 00 \end{array}$$

We now have that $(M, 1)$ is bisimilar to $(M', 10)$ w.r.t. the epistemic language for atom p and agents a, b, but that $(M, 1)$ is not bisimilar to $(M', 10)$ w.r.t. the epistemic language for atoms p, q and agents a, b. Therefore, $M, 1 \models \psi$ iff $M', 10 \models \psi$. On the other hand

[1] Alternatively, one can use more straightforwardly the $S5$ validity $(K_a\varphi_0 \wedge \hat{K}_a\varphi_1) \rightarrow (K_a\varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1))$

[2] A formula is in *normal form* when it is a conjunction of disjunctions of the form $\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \ldots \vee K_a\varphi_n$. Every formula in single-agent epistemic logic (*K45* and therefore also in) *S5* is equivalent to a formula in normal form [10]. A normal form may not exist for a multi-agent formula, e.g., it does not exist for K_aK_bp.

[3] The alternative is that ψ contains a *finite* number of atoms. Now it would be mighty hard to determine *which* other atoms apart from p, but this does not matter: the contradiction on which the proof of Proposition 10 is based, merely requires a 'fresh' atom not yet occurring in ψ. Note that we cannot exclude that *all* atoms in the parameter set P occur in ψ. In that case there would not be a fresh atom after all: the proof therefore requires an infinite set P. Barteld Kooi recently gave an alternative proof without that requirement.

$M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$ but, instead, $M', 10 \models \Diamond(K_a p \wedge \neg K_b K_a p)$. This is because $M', 10 \models \langle p \vee q\rangle(K_a p \wedge \neg K_b K_a p)$: the announcement $p \vee q$ restricts the domain to the three states where it is true, and $M'|(p \vee q), 10 \models K_a p \wedge \neg K_b K_a p$, because $10 \sim_b 11$ and $M'|(p \vee q), 11 \models \neg K_a p$. □

The counterexample used in the proof of Proposition 10 can be adjusted to show that *compactness fails for our logic*.

Proposition 11 Arbitrary announcement logic is not compact.

Proof Take the following infinite set of formulas:

$$\{[\theta](K_a p \to K_b K_a p) : \theta \in \mathcal{L}_{el}\} \cup \{\neg\Box(K_a p \to K_b K_a p)\}.$$

By the semantics of $\Box$, this set is obviously not satisfiable. But we will show that *any of its finite subsets is satisfiable* (which contradicts compactness). Let

$$\{[\theta_i](K_a p \to K_b K_a p) : 1 \le i \le n\} \cup \{\neg\Box(K_a p \to K_b K_a p)\}$$

be any such finite subset, and let q be an atomic sentence that is distinct from p and does not occur in any of the sentences θ_i ($1 \le i \le n$). Take now the epistemic state $(M', 10)$ as in the proof to the previous counterexample. As shown above, we have $M', 10 \models \Diamond(K_a p \wedge \neg K_b K_a p)$, and thus $M', 10 \models \neg\Box(K_a p \to K_b K_a p)$. On the other hand, for the epistemic state $(M, 1)$ as in the above proof, we have shown above that we have $M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, i.e. $M, 1 \models \Box(K_a p \to K_b K_a p)$. By the semantics of $\Box$, it follows that $M, 1 \models [\theta_i](K_a p \to K_b K_a p)$ for all $1 \le i \le n$; but q doesn't occur in any of these formulas, so their truth-values must be the same at $(M', 10)$ and $(M, 1)$ (since as shown above, the two epistemic states are bisimilar w.r.t. the language without q). Thus, we have $M', 10 \models [\theta_i](K_a p \to K_b K_a p)$ for all $1 \le i \le n$. Putting these together, we see that our finite set of formulas is satisfied at the state $(M', 10)$. □

3.3 KNOWABILITY

A suitable direction of research is the syntactic or semantic characterisation of interesting fragments of the logic. In this section we define *positive*, *preserved*, *successful*, and *knowable* formulas, and investigate their relation.

The *positive* $\mathcal{L}_{apal}$ formulas intuitively correspond to formulas that do not express ignorance, i.e., in epistemic logical ($\mathcal{L}_{el}$) terms: in which negations do not precede K_a operators. The fragment of the *positive formulas* is inductively defined as

$$\varphi ::= p|\neg p|\varphi \vee \varphi|\varphi \wedge \varphi|K_a\varphi|[\neg\varphi]\varphi|\Box\varphi$$

The negation in $[\neg\varphi]\varphi$ is there for technical reasons but unfortunately makes 'positive' somewhat of a misnomer.

The *preserved formulas* preserve truth under arbitrary (epistemically definable) model restriction (relativisation). They are (semantically) defined as those φ for which $\models \varphi \to \Box\varphi$. There is no corresponding semantic principle in public announcement logic that expresses truth preservation.

We will prove that positive formulas are preserved. Restricted to epistemic logic without common knowledge, this was observed by van Benthem in [14]. Van Ditmarsch and Kooi extended this in [16] to public announcement logic, with clause $[\neg\varphi]\varphi$ (and with common knowledge operators; however, without Van Benthem's characterisation result). Note that the truth of the announcement is a *condition* of its execution, which, when seen as a disjunction, explains the negation in $[\neg\varphi]$. Surprisingly, we can expand this fragment with $\Box\varphi$ for arbitrary announcement logic: in the case $\Box\varphi$ of the inductive proof below to show truth preservation, assuming the opposite easily leads to a contradiction.

Proposition 12 Positive formulas are preserved.

Proof For 'M' is a submodel of M' write $M' \subseteq M$. To prove the proposition it is sufficient to show that for arbitrary M, M' with $M' \subseteq M$ and $s \in \mathcal{D}(M')$, and arbitrary φ, if $(M, s) \models \varphi$, then $(M', s) \models \varphi$ (because it then also holds for all *epistemically definable* submodels M'). We show that by proving an even slightly stronger proposition, namely: "for arbitrary M, M', M'' with $M'' \subseteq M' \subseteq M$ and $s \in \mathcal{D}(M'')$, and arbitrary φ, if $(M', s) \models \varphi$, then $(M'', s) \models \varphi$," which has the advantage of loading the induction hypothesis. This is needed for the case $[\neg\varphi]\psi$ of the proof, that is by induction on the formula. We assume most cases to be well-known, except for $[\neg\varphi]\psi$, which we copy from [16], and $\Box\varphi$, which is new.

Case $[\varphi]\psi$:

Given is $(M', s) \models [\neg\varphi]\psi$. We have to prove that $(M'', s) \models [\neg\varphi]\psi$. Assume that $(M'', s) \models \neg\varphi$. By using the contrapositive of the induction hypothesis, $(M', s) \models \neg\varphi$. From that and the assumption $(M', s) \models [\neg\varphi]\psi$ follows $(M'|\neg\varphi, s) \models \psi$. Because $(M', s) \models \neg\varphi$, $M''|\neg\varphi$ is a submodel of $M'|\neg\varphi$. From $(M'|\neg\varphi, s) \models \psi$ and $M''|\neg\varphi \subseteq M'|\neg\varphi \subseteq M' \subseteq M$ it follows from (the loaded version of!) induction that $(M''|\neg\varphi, s) \models \psi$. Therefore $(M'', s) \models [\neg\varphi]\psi$.

Case $\Box\varphi$:

Assume $(M, s) \models \Box\varphi$. Suppose towards a contradiction that $(M'', s) \not\models \Box\varphi$. Then there is a ψ such that $(M'', s) \models \langle\psi\rangle\neg\varphi$, i.e. $(M''|\psi, s) \not\models \varphi$. From $M''|\psi \subseteq M'' \subseteq M$ and contraposition of induction follows $(M, s) \not\models \varphi$. But from $(M, s) \models \Box\varphi$ follows $(M, s) \models [\top]\varphi$ which equals $(M, s) \models \varphi$ that contradicts the previous. $\Box$

We do not know whether the preserved formulas are (logically equivalent to) positive. An answer to this question seems hard. It would extend Van Benthem's results in [14].

Another semantic notion is that of *success*. *Successful formulas* are believed after their announcement, or, in other words, after 'revision' with that formula. This precisely corresponds to the postulate of 'success' in AGM belief revision. Formally, φ is a *successful formula* iff $[\varphi]\varphi$ is valid (see [16], elaborating an original but slightly different proposal in [5]). The validity of $[\varphi]\varphi$ corresponds to the validity of $\varphi \to [\varphi]K_a\varphi$ [16]: announced formulas be believed after their true announcement.

Proposition 13 Preserved formulas are successful.

Proof $\models \varphi \to \Box\varphi$ implies $\models \varphi \to [\varphi]\varphi$, and $\models \varphi \to [\varphi]\varphi$ iff $\models [\varphi]\varphi$. $\Box$

Corollary 14 Positive formulas are successful.

Fitch investigated the formulas that, if true, can become known [3]. Consider a multi-agent version. We define the *knowable formulas* as those for which, for all agents $a \in A$, $\models \varphi \to \Diamond K_a\varphi$. We can now observe that, e.g.:

Proposition 15 Positive, preserved, and successful formulas are all knowable.

Proof Similar to the proof of Prop. 13. Observe that $\varphi \to \Box\varphi$ implies $\varphi \to [\varphi]K_a\varphi$; $\varphi \to [\varphi]K_a\varphi$ implies $\varphi \to \langle\varphi\rangle K_a\varphi$; and $\varphi \to \langle\varphi\rangle K_a\varphi$ implies $\varphi \to \Diamond K_a\varphi$. $\Box$

We did not investigate the knowable formulas in depth. Some knowable formulas are not positive, for example $\neg K_a p$: if true, announce $\top$, and $K_a\neg K_a p$ (still!) holds. Therefore $\models \neg K_a p \to \Diamond K_a \neg K_a p$.

4 AXIOMATISATION

We now provide a complete axiomatisation of $\mathcal{L}_{apal}$. We apply a technique suggested by Goldblatt [7] using 'necessity forms'. A *necessity form* contains a unique occurrence of a special symbol $\sharp$. If $\boldsymbol{\psi}$ is such a necessity form (we write boldface Greek letters for arbitrary necessity forms) and $\varphi \in \mathcal{L}_{apal}$, then $\boldsymbol{\psi}(\varphi)$ is obtained from $\boldsymbol{\psi}$ by substituting $\sharp$ in $\boldsymbol{\psi}$ for φ.

The *necessity forms* are inductively defined as follows. Let $\varphi \in \mathcal{L}_{apal}$. Then:

- $\sharp$ is a necessity form,
- if $\boldsymbol{\psi}$ is a nec. form then $(\varphi \to \boldsymbol{\psi})$ is a nec. form,
- if $\boldsymbol{\psi}$ is a necessity form then $[\varphi]\boldsymbol{\psi}$ is a nec. form,
- if $\boldsymbol{\psi}$ is a necessity form then $K_a\boldsymbol{\psi}$ is a nec. form.

Necessity forms are used in the derivation rule $R(\Box)$ of the axiomatisation **APAL**, now to follow.

Definition 16 The axiomatisation **APAL** is given in Table 1. A formula is a *theorem* if it belongs to the least set of formulae containing all axioms and closed under the rules. If φ is a theorem, we write $\vdash \varphi$.

instant. of prop. tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distr. of kn. over impl.
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	pos. introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	neg. introspection
$[\varphi]p \leftrightarrow (\varphi \to p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	ann. and negation
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	ann. and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	ann. and knowledge
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	ann. composition
$\Box\varphi \to [\psi]\varphi$ where $\psi \in \mathcal{L}_{el}$	arb. and specific ann.
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	nec. of knowledge
From φ, infer $[\psi]\varphi$	nec. of announcement
From φ, infer $\Box\varphi$	nec. of arb. ann.
From $\boldsymbol{\varphi}([p]\psi)$, infer $\boldsymbol{\varphi}(\Box\psi)$ where $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$	deriving arb. ann. (a.k.a. $R(\Box)$)

Table 1: The axiomatisation **APAL**

All axioms and rules are sound. We only pay attention to the rules and axioms involving $\Box$. Proposition 5.4 proved soundness for 'necessitation of arbitrary announcement'. In the axiom 'arbitrary and specific announcement', the restriction to epistemic formulas is important. Without that restriction the axiom is unsound. Also the rule $R(\Box)$ is correct with respect to the semantics. We sketch the proof. Suppose, towards a contradiction, that $\boldsymbol{\varphi}([p]\psi)$ is valid but $\boldsymbol{\varphi}(\Box\psi)$ is *not* valid, i.e. we have a model such that $(\mathbf{S}, V, s) \models \neg\boldsymbol{\varphi}(\Box\psi)$. By going to the dual form ('possibility form'), and by applying an omitted Lemma stating that a $\Diamond$ operator in a possibility form can be 'simulated' by announcing a fresh atom p (i.e., by some $\langle p\rangle$), we derive that $(\mathbf{S}, V', s) \models \neg\boldsymbol{\varphi}([p]\psi)$ where V' only differs from V for atom p. This contradicts the validity of $\boldsymbol{\varphi}([p]\psi)$.

Example 17 We show that the validity $\Box\varphi \to \Box\Box\varphi$ is also a theorem. In step 2 of the derivation we are informal, as the placeholder $q \wedge [q]r$ for $\Box$ is not an $\mathcal{L}_{el}$ formula. The informality is justified because each $\mathcal{L}_{pal}$ is equivalent to a $\mathcal{L}_{el}$ formula. (Formally, several more steps are required, where we use that subformula $[q]r$ in $(q \wedge [q]r)$ is equivalent to $q \to r$.) In step 4 of the derivation we use that $\Box p \to [q]\sharp$ is a necessity form, and in step 5 of the derivation we use that $\Box p \to \sharp$ is a necessity form.

1. $\vdash [q \wedge [q]r]p \leftrightarrow [q][r]p$	ann. composition
2. $\vdash \Box p \to [q \wedge [q]r]p$	arbitrary and specific ann.
3. $\vdash \Box p \to [q][r]p$	1, 2, propositionally
4. $\vdash \Box p \to [q]\Box p$	3, $R(\Box)$
5. $\vdash \Box p \to \Box\Box p$	4, $R(\Box)$

Example 18 For another example, note that regardless of the restriction that $\psi \in \mathcal{L}_{el}$, in $\Box\varphi \to [\psi]\varphi$ (arbitrary and specific announcement), there are already very basic theorems of the form $[\psi]\varphi$ where ψ is *not* an epistemic formula. In other words: this restriction is not 'per se' a reason to fear incompleteness of the logic.

1. $\vdash \Box p \to [\top]p$	arb. and specific announcement
2. $\vdash [\top]p \to (\top \to p)$	atomic permanence
3. $\vdash (\top \to p) \leftrightarrow p$	propositionally
4. $\vdash \Box p \to p$	1, 2, 3, propositionally
5. $\vdash [\Box p]p \leftrightarrow (\Box p \to p)$	atomic permanence
6. $\vdash [\Box p]p$	4, 5, propositionally

Using Goldblatt's technique applying necessity forms, one can now prove completeness for the logic in the standard way. The main effect of rule $R(\Box)$ is that it makes the canonical model (consisting of all maximal consistent sets of formulae closed under the rule) standard for $\Box$. Let us see how.

Consider a variant of the axiomatisation with an infinitary rule $R^\omega(\Box)$ formulated as follows:

- from $\boldsymbol{\varphi}([\chi]\psi)$ for all $\chi \in \mathcal{L}_{el}$, infer $\boldsymbol{\varphi}(\Box\psi)$.

The reader may easily verify that this rule is sound. Our completeness proof for the finitary axiomatisation is indirect.

First, let us observe that the rule $R(\Box)$ is stronger than the rule $R^\omega(\Box)$: if we can prove $\boldsymbol{\varphi}([\theta]\psi)$ for all epistemic formulas θ then we can prove in particular $\boldsymbol{\varphi}([p]\psi)$ for some atom $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$. As a result, we can derive the conclusion of the infinitary rule using only the finitary rule, and the axiomatisation based on the infinitary rule $R^\omega(\Box)$ defines the same set of theorems as the axiomatisation based on the finitary rule $R(\Box)$.

Second, let us demonstrate that the axiomatisation based on the infinitary rule $R^\omega(\Box)$ is complete with respect to the semantics. A set x of formulas is called a theory if it satisfies the following conditions:

- x contains the set of all theorems;
- x is closed under the rule of modus ponens and the rule $R^\omega(\Box)$.

Obviously, the least theory is the set of all theorems whereas the greatest theory is the set of all formulas. The latter theory is called the trivial theory. A theory x is said to be consistent if $\bot \notin x$. Let us remark that the only inconsistent theory is the set of all formulas. We shall say that a theory x is maximal if for all formulas φ, $\varphi \in x$ or $\neg\varphi \in x$. Let x be a set of formulas. For all formulas φ, let $x+\varphi = \{\psi\colon \varphi \to \psi \in x\}$. For all agents a, let $K_a x = \{\varphi\colon K_a\varphi \in x\}$. For all formulas φ, let $[\varphi]x = \{\psi\colon [\varphi]\psi \in x\}$.

Lemma 19 Let x be a theory, φ be a formula, and a be an agent. Then $x + \varphi$, $K_a x$ and $[\varphi]x$ are theories. Moreover $x + \varphi$ is consistent iff $\neg\varphi \notin x$.

Proof We will only prove that $K_a x$ is a theory. First, let us prove that $K_a x$ contains the set of all theorems. Let ψ be a theorem. By the necessitation of knowledge, $K_a\psi$ is also a theorem. Since x is a theory, then $K_a\psi \in x$. Therefore, $\psi \in K_a x$. It follows that $K_a x$ contains the set of all theorems. Second, let us prove that $K_a x$ is closed under modus ponens. Let ψ, χ be formulas such that $\psi \in K_a x$ and $\psi \to \chi \in K_a x$. Thus, $K_a\psi \in x$ and $K_a(\psi \to \chi) \in x$. Since $K_a\psi \to (K_a(\psi \to \chi) \to K_a\chi)$ is a theorem and x is a theory, then $K_a\psi \to (K_a(\psi \to \chi) \to K_a\chi) \in x$. Since x is closed under modus ponens, then $K_a\chi \in x$. Hence, $\chi \in K_a x$. It follows that $K_a x$ is closed under modus ponens. Third, let us prove that $K_a x$ is closed under $R^\omega(\Box)$. Let $\boldsymbol{\varphi}$ be a necessity form and ψ be a formula such that $\boldsymbol{\varphi}([\chi]\psi) \in K_a x$ for all $\chi \in \mathcal{L}_{el}$. It follows that $K_a\boldsymbol{\varphi}([\chi]\psi) \in x$ for all $\chi \in \mathcal{L}_{el}$. Since x is a theory, then $K_a\boldsymbol{\varphi}(\Box\psi) \in x$. Consequently, $\boldsymbol{\varphi}(\Box\psi) \in K_a x$. It follows that $K_a x$ is closed under $R^\omega(\Box)$. $\Box$

Lemma 20 (Lindenbaum lemma) Let x be a consistent theory. There exists a maximal consistent theory y such that $x \subseteq y$.

Proof Let $\psi_0, \psi_0, \ldots$ be a list of the set of all formulas. We define a sequence $y_0, y_1, \ldots$ of consistent theories as follows. First, let $y_0 = x$. Second, suppose that, for some $n \geq 0$, y_n is a consistent theory containing x that has been already defined. If $y_n + \psi_n$ is inconsistent and $y_n + \neg\psi_n$ is inconsistent then, by lemma 19,

$\neg\psi_n \in y_n$ and $\neg\neg\psi_n \in y_n$. Since $\neg\psi_n \to (\neg\neg\psi_n \to \bot)$ is a theorem, then $\neg\psi_n \to (\neg\neg\psi_n \to \bot) \in y_n$. Since y_n is closed under modus ponens, then $\bot \in y_n$: a contradiction. Hence, either $y_n + \psi_n$ is consistent or $y_n + \neg\psi_n$ is consistent. If $y_n + \psi_n$ is consistent then we define $y_{n+1} = y_n + \psi_n$. Otherwise, $\neg\psi_n \in y_n$ and we consider two cases:

In the first case, we suppose that ψ_n is not a conclusion of $R^\omega(\Box)$. Then, we define $y_{n+1} = y_n$.

In the second case, we suppose that ψ_n is a conclusion of $R^\omega(\Box)$. Let $\boldsymbol{\varphi_1}(\Box\chi_1)$, ..., $\boldsymbol{\varphi_k}(\Box\chi_k)$ be all the representations of ψ_n as a conclusion of $R^\omega(\Box)$. We define the sequence $y_n^0, \ldots, y_n^k$ of consistent theories as follows. First, let $y_n^0 = y_n$. Second, suppose that, for some $i < k$, y_n^i is a consistent theory containing y_n that has been already defined. Then it contains $\neg\boldsymbol{\varphi_i}(\Box\chi_i)$. Since y_n^i is closed under $R^\omega(\Box)$, then there exists a formula $\varphi_i \in \mathcal{L}_{el}$ such that $\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$ is not in y_n^i. Then, we define $y_n^{i+1} = y_n^i + \neg\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$. Now, we put $y_{n+1} = y_n^k$. Finally, we define $y = y_0 \cup y_1 \cup \ldots$. It is straightforward to prove that y is a maximal consistent theory such that $x \subseteq y$. □

The canonical model of $\mathcal{L}_{apal}$ is the structure $\mathcal{M}_c = (W, \sim, V)$ defined as follows:

- W is the set of all maximal consistent theories;
- For all agents a, $\sim_a$ is the binary (equivalence) relation on W defined by $x \sim_a y$ iff $K_a x = K_a y$;
- For all atoms p, V_p is the subset of W defined by $x \in V_p$ iff $p \in x$.

Lemma 21 (Truth lemma) Let φ be a formula in $\mathcal{L}_{apal}$. Then for all maximal consistent theories x and for all finite sequences $\vec{\psi} = (\psi_1, \ldots, \psi_k)$ of formulas in $\mathcal{L}_{apal}$ such that $\psi_1 \in x$, $[\psi_1]\psi_2 \in x$, ..., $[\psi_1]\ldots[\psi_{k-1}]\psi_k \in x$:

$$\mathcal{M}_c|\vec{\psi}, x \models \varphi \text{ iff } [\psi_1]\ldots[\psi_k]\varphi \in x.$$

Proof The proof is by induction on φ. The base case follows from the definition of V. The Boolean cases are trivial. It remains to deal with the modalities. We only present the case of the epistemic modality. If $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$ then there exists a maximal consistent theory y such that $x \sim_a y$, $\psi_1 \in y$, $[\psi_1]\psi_2 \in y$, ..., $[\psi_1]\ldots[\psi_{k-1}]\psi_k \in y$ and $\mathcal{M}_c|\vec{\psi}, y \not\models \varphi$. By induction hypothesis, $[\psi_1]\ldots[\psi_k]\varphi \notin y$. Since $x \sim_a y$, then $K_a x = K_a y$. Thus, $[\psi_1]\ldots[\psi_k]K_a\varphi \notin x$. Reciprocally, if $[\psi_1]\ldots[\psi_k]K_a\varphi \notin x$ then $K_a[\psi_1]\ldots[\psi_k]\varphi \notin x$. Let $y = K_a x + \neg[\psi_1]\ldots[\psi_k]\varphi$. The reader may easily verify that y is a consistent theory. By Lindenbaum lemma, there is a maximal consistent theory z such that $y \subseteq z$. Hence, $K_a x \subseteq z$ and $[\psi_1]\ldots[\psi_k]\varphi \notin z$. Consequently, $x \sim_a z$, $\psi_1 \in z$, $[\psi_1]\psi_2 \in z$, ..., $[\psi_1]\ldots[\psi_{k-1}]\psi_k \in z$ and, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}, z \not\models \varphi$. Therefore, $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$. □

As a result we have:

Theorem 22 (Soundness and completeness) Let φ be a formula. Then φ is a theorem iff φ is valid.

Proof Soundness is immediate, following the observations at the beginning of this section. Completeness follows from Lemmas 19, 20, and 21. □

5 FURTHER WORK

We proposed an extension of public announcement logic with a dynamic modal operator $\Box\varphi$ expressing that φ is true after every announcement ψ. We gave various semantic results, defined fragments of 'knowable' formulas in the Fitch sense that $\models \varphi \to \Diamond K_a\varphi$, and we showed completeness for a Hilbert-style axiomatisation of this logic. There is much more work to do, and we already have various additional results not reported in this conference contribution.

The model checking problem for the logic $APAL$ is decidable under the fairly strong restrictions that models are finite and that only a finite number of atoms changes value in a given model. This result is not trivial, because of the implicit quantification over *all* atoms in the $\Box$-operator. We hope to obtain more results.

Along a common line in dynamic epistemics, one might expand the language with additional modal operators, in particular: with common knowledge, with actions that are not public (such as private announcements), and with assignments (actions that change the truth value of atomic propositions). Let us consider 'arbitrary actions' in the sense of arbitrary action models:

Let U be a finite action model [1]. Three possible generalisations are as follows ($*$ is arbitrary finite iteration). A sensible restriction in the semantics for arbitrary actions is that all preconditions must be epistemic formulas.

1	$M, s \models \langle U \rangle \varphi$	iff	$\exists u \in U : M, s \models \langle U, u \rangle \varphi$
2	$M, s \models \Diamond\varphi$	iff	$M, s \models \langle U \rangle^* \varphi$ for a given U
3	$M, s \models \Diamond\varphi$	iff	$\exists U : M, s \models \langle U \rangle \varphi$

The first was investigated by Hoshi in [9, p.8]. The second (where an action model U, or an action model frame, is a parameter for the language) can be seen as a generalisation of iterated relativisation which was investigated in [11]—this version is not promising if one is after decidable logics. The third *seems* a rather wild generalisation but is in fact very promising: for this operator all validities in Proposition 5 hold, and CR seems to hold as well, but MK does not hold: there certainly are infinite chains of informative actions.

If factual change is also permitted, one has a peculiar result that $\Diamond\varphi$ is valid for all consistent φ, in other words, all satisfiable formulas are realisable (reachable) in *any information state*.[4] Allowing factual change may be a too drastic departure from the original Fitch question which true formulas are knowable.

There are various other multi-agent versions of knowability that we would like to explore. For example, for which φ does it hold that knowledge is *transferable between agents*: $K_a\varphi \rightarrow \Diamond K_b\varphi$? Or for which φ does it hold that *distributed knowledge can be made common*: $D_A\varphi \rightarrow \Diamond C_A\varphi$?

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