Games, Geometry, and the Computational Complexity of Finding Equilibria

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Game-theoretic problems have found massive recent interest in computer science. Obvious applications arise from the internet and electronic commerce, for example the study of online auctions.

In theoretical computer science, some of the most intriguing open problems concern the time that an algorithm needs to find a Nash equilibrium of a game. In this talk, I give an introduction to these questions for non-experts in computational complexity.

In the first half of the talk, I explain the problem of solving zero-sum "simple stochastic games", which belong to the complexity class "NP ∩ co-NP". This class contains famous problems like primality of an integer, or linear programming, for which polynomial-time algorithms have been found after long research, which were breakthrough results.

The simple stochastic games in question are defined by a directed graph with two special sinks called 0 and 1. All other nodes have two outgoing arcs, and are either "Average" nodes or belong to the Max player or to the Min player. From a given starting node, a token is chased around the graph until it reaches one of the sinks, with the corresponding payoff 0 or 1 paid by the Min player to the Max player. The next node is chosen randomly with probability $1/2$ from an Average node, or by the Max or the Min player if the token is on a node of that player. The Max player tries to maximize and Min player tries to minimize the payoff to Max, which is the probability of reaching the 1 sink. The structure of the graph is such that irrespective of the player's choices, a sink is eventually reached with probability one. In particular, there are no cycles that exclusively belong to only Max or Min.

It can be shown that there exist pure and history-independent optimal strategies of the players. Optimality is easily verified by the expected payoff assigned to each node of the graph for the given strategy pair. This shows that the decision problem whether the value of the game is at least some number v in $[0, 1]$ belongs to the complexity class NP ∩ co-NP.

The computational problem of solving simple stochastic games was posed in this way by Condon (1992). Despite much effort, a polynomial-time algorithm continues to be elusive.

The second half of the talk is concerned with the problem of finding one Nash equilibrium of a bimatrix game, that is, a two-player game in strategic form specified by two integer payoff matrices. A recent result (Chen and Deng (2005), not discussed in the talk), shows that this problem is complete for the complexity class PPAD, which includes a computational version of Brouwer's fixed point theorem, a seemingly much harder problem. An algorithm for finding one Nash equilibrium of a bimatrix game in subexponential time would already be a major breakthrough in the field.

I present joint work with Rahul Savani (Savani and von Stengel (2006)) that shows that finding an equilibrium can take exponential time when using the well known algorithm by Lemke and Howson (1964). The algorithm is a pivoting method similar to the simplex algorithm for linear programming. The simplex algorithm has been shown to be exponential by Klee and Minty (1972), but this question was open for the Lemke–Howson algorithm.

We give a class of square bimatrix games for which the shortest Lemke–Howson path grows exponentially in the dimension d of the game. As will be explained, equilibria of a bimatrix game are described by a combinatorial condition on vertices of two polytopes given by the payoff matrices. The games are constructed using pairs of dual cyclic polytopes with $2d$ facets in d-space, a classic object in polyhedral geometry. The lengths of the Lemke–Howson paths are described by a Fibonacci sequence and therefore grow exponentially.

The talk is directed at a general audience.

References

X. Chen and X. Deng (2005), Settling the complexity of 2-player Nash-equilibrium. *Electronic Colloquium on Computational Complexity* TR05-140.

A. Condon (1992), The complexity of stochastic games. *Information and Computation* 96, 203–224.

V. Klee and G. J. Minty (1972), How good is the simplex algorithm? In: *Inequalities, III*, Proc. Third Symp., UCLA, 1969, ed. O. Shisha, Academic Press, New York, 159–175.

C. E. Lemke and J. T. Howson, Jr. (1964), Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics* 12, 413–423.

R. Savani and B. von Stengel (2006), Hard-to-solve bimatrix games. *Econometrica* 74, 397–429.