# Robust Equilibria under Non-Common Priors*

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## Abstract

This paper considers the robustness of equilibria to a small amount of incomplete information, where players are allowed to have heterogenous priors. An equilibrium of a complete information game is *robust to incomplete information under non-common priors* if for every incomplete information game where each player's prior puts high probability on the event that the players know at arbitrarily high order that the payoffs are given by the complete information game, there exists a Bayesian Nash equilibrium that generates behavior close to the equilibrium in consideration. It is shown that for generic games, an equilibrium is robust under non-common priors if and only if it consists of the unique rationalizable action profile. Set valued concepts are also introduced, and for generic games, a smallest robust set is shown to exist and coincide with the set of a posteriori equilibria.

## 1 INTRODUCTION

One important research program in game theory has been to examine the robustness of a Nash equilibrium of a given complete information game to incomplete information, i.e., whether the predictions generated by the Nash equilibrium are still good predictions in "nearby" incomplete information games obtained by perturbing the complete information game (see, e.g., Fudenberg, Kreps, and Levine (1988) and Kajii and Morris (1997)). There, most existing approaches (Kajii and Morris (1997), Ui (2001), and Morris and Ui (2005), among others) assume that players share a common prior in perturbed incomplete information games, as do most work in other fields in game theory and information economics. In this paper, we characterize equilibria that remain good predictions in perturbed incomplete information games dropping the common prior assumption (CPA, henceforth), i.e., allowing players to have heterogeneous subjective prior beliefs. This enables us to assess the role of the CPA in examining the robustness of equilibria to incomplete information.

To explain our framework, consider an analyst who plans to model some strategic situation by a particular complete information game. While he believes that the environment is described by this game with high probability, he is also aware that there is a small amount of payoff uncertainty, so that the players may play some incomplete information game that is close to the complete information game, where he does not assume that the players share a common prior. We want to ask whether the analyst's prediction based on the complete information game is not qualitatively different from some equilibrium of the real incomplete information game.

Our key assumption to formalize closeness between incomplete information games and a complete information game $\mathbf{g}$ is that the analyst is restricted to incomplete information games where agents know that the game is $\mathbf{g}$ only up to finitely many orders of knowledge. To be more precise, the complete information situation assumes that it is common knowledge among players that the game played is $\mathbf{g}$. Intuitively speaking, this says that everyone knows that the game is $\mathbf{g}$ (the game is mutual knowledge), everyone knows that everyone knows that the game is $\mathbf{g}$ (the game is mutual knowledge at order two), and so on. In our setting, the analyst does not know the entire hierarchy of knowledge. Indeed, he is confident in his model up to a certain extent so that he believes that with a high probability players mutually know that the real game is $\mathbf{g}$ up to a finite level (possibly very large). To be specific, fix an equilibrium of a complete information game $\mathbf{g}$. An incomplete information game is an $(\varepsilon, N)$-perturbation if the action sets are same as those of $\mathbf{g}$ and each player's prior puts probability at least $1-\varepsilon$ on the event that the players know at order $N$ that the payoffs are given by $\mathbf{g}$. An equilibrium is *robust to incomplete information under non-common priors* in $\mathbf{g}$ if there

*This is an extract from the paper with the same title. The full paper is available at www.econ.hit-u.ac.jp/˜oyama/papers/rbstNCP.html.

exist $\varepsilon > 0$ and $N \geq 0$ such that every $(\varepsilon, N)$-perturbation of $\mathbf{g}$ has a Bayesian Nash equilibrium [1] under which the ex-ante probability that each player assigns to any action profile is approximately given by the equilibrium of $\mathbf{g}$. This has the important implication that under this Bayesian Nash equilibrium the ex-ante (subjective) payoffs of each player is approximately given by the equilibrium payoffs in $\mathbf{g}$.[2]

*Our first main result* shows that for generic games, a Nash equilibrium is robust under non-common priors in $\mathbf{g}$ if and only if it is the unique rationalizable action profile of $\mathbf{g}$. Its sufficiency follows from the assumption that in incomplete information perturbations close to $\mathbf{g}$, the real game is mutually known to be $\mathbf{g}$ up to high enough order (at least the number of the iteration rounds needed to reach the single action profile). To show the necessity, which is the main part of this paper, we obtain a contagion result for a posteriori equilibria:[3] for any a posteriori equilibrium of a generic game and for any $\varepsilon > 0$ and $N \geq 0$, we construct a dominance solvable $(\varepsilon, N)$-perturbation whose unique rationalizable strategy profile generates an action distribution that can be arbitrarily close to this a posteriori equilibrium. From the result by Brandenburger and Dekel (1987), we know that if more than one action profile is rationalizable, then there are several a posteriori equilibria. Hence, if more than one action survive iterative elimination of actions that are never best responses, then no action profile is robust.

Brandenburger and Dekel (1987) show that for any a posteriori equilibrium of a given complete information game, we can add payoff-irrelevant types to have an incomplete information game with non-common priors whose Bayesian Nash equilibrium generates the distribution of the a posteriori equilibrium. In contrast, our contagion result used for our necessity result shows that (in generic games) when we allow for payoff-relevant types that have vanishingly small prior probability, the above Bayesian Nash equilibrium can indeed be the unique rationalizable strategy profile of a dominance solvable incomplete information game. We note that it is crucial for our result as well as for the result of Brandenburger and Dekel (1987) to drop the CPA.

Since many games have no robust equilibrium under non-common priors, it is natural to ask if a set of equilibria is robust. Indeed, Kohlberg and Mertens (1986) propose making set of equilibria the object of a theory of equilibrium refinements. Following their program as well as Morris and Ui's (2005), we also investigate the robustness of set of equilibria. A set of equilibria of a complete information game is *robust to incomplete information under non-common priors* if there exist $\varepsilon > 0$ and $N \geq 0$ such that any $(\varepsilon, N)$-perturbation has a Bayesian Nash equilibrium whose behavior can be approximated by some action distribution in this set.[4] If a robust set is a singleton, then the equilibrium is robust in our previous sense. A set of action distributions is called a smallest robust set if it is robust and is contained in any robust set. *Our second main result* (and our most general result) shows that for generic games, a smallest robust set exists and coincides with the the set of a posteriori equilibria.

Kajii and Morris (1997) introduce the notion of robustness of equilibria to incomplete information under common prior. They consider incomplete information perturbations of a given complete information where the players share a common prior. They show in particular that a $p$-dominant equilibrium[5] with $p$ sufficiently small is robust to incomplete information under common priors. Following Kajii and Morris (1997), papers by Ui (2001), Morris and Ui (2005), and Oyama and Tercieux (2004) provide sufficient conditions for a Nash equilibrium to be robust to incomplete information under common prior. Our result shows that when we relax the common prior assumption, none of the existing sufficient conditions implies robustness under non-common priors.

Weinstein and Yildiz (2007) consider a notion of *interim robustness*. A Nash equilibrium $a^*$ is interim robust in $\mathbf{g}$ if for some $N \geq 0$ for any incomplete information game with (or without) common prior where the action sets are same as those of $\mathbf{g}$, there exists a Bayesian Nash equilibrium, say $\sigma$, such that in any state of the world where it is mutually known up to order $N$ that $\mathbf{g}$ is the true game, $a^*$ is played under $\sigma$. They show that for generic games, a Nash equilibrium is interim robust in $\mathbf{g}$ if and only if it the unique rationalizable action profile of $\mathbf{g}$. Contrary to that for our robustness concept, this characterization remains unchanged even if we restrict our attention to incomplete information games with common prior. This result follows from a result of Lipman (2003, 2005), which says that given any partition model (with a finite state space) and any state of the world in the model, for any finite $N$ there is a partition model with a common prior and a state in that model at which all the same facts about the world as well as all the same statements about beliefs and knowledge of order less than $N$ are true. Thus, in an interim context where the analyst has to make a prediction after players have received their private information (but before they take actions), imposing the CPA does not alter the set of robust predictions.

[1] We will actually prove that our results are unchanged when the equilibrium concept considered is given by any non-empty refinement of (interim) rationalizability.

[2] We choose a formulation of our robustness test in terms of equilibria rather than in terms of equilibrium payoffs to make easier the comparison with the previous literature (in particular with Kajii and Morris (1997)).

[3] An a posteriori equilibrium is a refinement of subjective correlated equilibrium introduced and studied in Aumann (1974) and Brandenburger and Dekel (1987).

[4] As noted previously for the point-valued test, this robustness test can be written in terms of sets of equilibrium payoffs rather than sets of equilibria.

[5] See Morris, Rob, and Shin (1995) and Kajii and Morris (1997).

On the other hand, we conclude that in the context where there exists an ex ante stage in which the analyst is interested in the ex ante behavior, he may need to know how players behave at each state of the world in his model and he may not be interested in the behavior at a particular single state. In this case, the common prior assumption has a real bite and allowing for models with heterogeneous priors has important (strategic) consequences.

To prove their main result Weinstein and Yildiz (2007) show that for any complete information type in the universal type space[6] (see Mertens and Zamir (1985) and Brandenburger and Dekel (1993)) and any rationalizable action profile $a^*$ of this game, there exists a dominance-solvable incomplete information game and a sequence of types drawn from this game such that (1) this sequence converges to the complete information type (with respect to the product topology in the universal type space) (2) each type of the sequence plays $a^*$.[7]

To establish our results we show that the dominance-solvable incomplete information game can be an $(\varepsilon, N)$-perturbation (where $\varepsilon$ can be arbitrarily small and $N$ arbitrarily large) and in the case where it is an $(\varepsilon, N)$-perturbation we fully characterize the unique equilibrium of this game using the notion of a posteriori equilibria (this is what we called earlier our contagion result). Whenever $a^*$ is a strict Nash equilibrium, the unique equilibrium of the dominance-solvable game may play $a^*$ everywhere on the type space, however, it is worth noting that if $a^*$ is not a strict Nash equilibrium, this is not possible: $a^*$ cannot be played everywhere[8]. A new contagion argument is needed to prove our results which relies on the notion of a posteriori equilibrium.

The point behind our result is that under non-common priors, a small (ex ante) probability event can have a larger impact on higher order beliefs than under common prior. The "critical path result" of Kajii and Morris (1997, Proposition 4.2) shows that under common prior, the impact of a small probability event is not large enough in the sense that, in some games and for some strict Nash equilibrium, a small amount of payoff uncertainty cannot induce this equilibrium to be played everywhere on the state space (i.e., it is not contagious). For instance, in $2 \times 2$ coordination games, the risk-dominated equilibrium cannot be contagious. In a companion paper (Oyama and Tercieux (2005)), we demonstrate, in contrast, that with non-common priors, any strict Nash equilibrium can be contagious. In that paper, for two-player incomplete information games with non-common priors, we model the strategic impact of an event by the notion of belief potential (Morris, Rob and Shin (1995)). We find the measure of discrepancy from the CPA so that the belief potential of an event has an upper bound that is an increasing function of this measure. Indeed, in order to have any strict Nash equilibrium to spread, this measure of discrepancy has to be large. In the present paper, we extend this observation and show that for any a posteriori equilibrium of any complete information game to be induced by a unique rationalizable strategy of some dominance solvable incomplete information perturbation, the ratio among prior probabilities in these perturbations need to be arbitrarily large.[9]

The remainder of the paper is organized as follows. Section 2 presents our notions of nearby incomplete information games and robustness. Section 3 states and proves our characterization of robust equilibria.

# 2 FRAMEWORK

## 2.1 COMPLETE INFORMATION GAMES

A complete information game consists of the set of players, $\mathcal{I} = \{1, 2, \ldots, I\}$, the finite set of actions, $A_i$, for each player $i \in \mathcal{I}$, and the payoff function, $g_i \colon A \to R$, for each player $i \in \mathcal{I}$. Throughout our analysis, we fix a complete information game, simply denoted by $\mathbf{g} = (g_i)_{i \in I}$.

For any at most countable set $S$, we denote by $\Delta(S)$ the set of all probability measures on $S$. We call elements in $\Delta(A)$ action distributions. For $a \in A$, we write $[a]$ for the element in $\Delta(A)$ that assigns weight one to $a$. For $\xi \in \Delta(A)$ and $a_i \in A_i$, we denote $\xi(a_i) = \sum_{a_{-i} \in A_i} \xi(a_i, a_{-i})$, and if $\xi(a_i) > 0$, we define $\xi(\cdot|a_i) \in \Delta(A_{-i})$ by $\xi(a_{-i}|a_i) = \xi(a_i, a_{-i})/\xi(a_i)$. We endow $\Delta(A)$ with the sup (or max) norm: $|\xi| = \max_{a \in A} \xi(a)$ for $\xi \in \Delta(A)$. For $\delta > 0$, we denote $V_\delta(\xi) = \{\xi' \in \Delta(A) \mid |\xi' - \xi| < \delta\}$ for $\xi \in \Delta(A)$ and $V_\delta(\Xi) = \{\xi' \in \Delta(A) \mid |\xi' - \xi| < \delta \text{ for some } \xi \in \Xi\}$ for $\Xi \subset \Delta(A)$. For $\mu = (\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$, we also denote $|\mu| = \max_{i \in \mathcal{I}} |\mu_i|$.

Given $\mathbf{g}$, let $br_i \colon \Delta(A_{-i}) \to A_i$ be the best response correspondence in pure actions for player $i \in \mathcal{I}$:

$$br_i(\pi_i) = \arg\max_{a_i \in A_i} g_i(a_i, \pi_i)$$

[6]A complete information type is a (degenerate) type in the universal type space where it is common knowledge that payoffs are given by the complete information game. Note that Weinstein and Yildiz (2007) do not restrict their attention to complete information types.

[7]In this context, where we consider only complete information types, it is easy to show that this incomplete information game can be both dominance-solvable and satisfying the common prior assumption. However, if we require both dominance-solvability and that players share a common prior, this incomplete information game need not be an $(\varepsilon, N)$-perturbation for $\varepsilon$ small and $N$ large. See Oyama and Tercieux (2005).

[8]More precisely, to have contagion of $a^*$, action profiles different from $a^*$ need to be contagious as well.

[9]If we first fix a given a posteriori equilibrium of a complete information game, then we can find a finite lower bound of the ratio for the incomplete information perturbation to generate the a posteriori equilibrium. Note that the same comment applies to the result of Brandenburger and Dekel (1987).

for $\pi_i \in \Delta(A_{-i})$, where $g_i(a_i, \cdot)$ is extended to $\Delta(A_{-i})$ in the usual way. We define *correlated rationalizability* (e.g., Brandenburger and Dekel (1987)). For each $i \in \mathcal{I}$, set $S_i^0[\mathbf{g}] = A_i$. Then, for $k = 1, 2, \ldots$, define $S_i^k[\mathbf{g}]$ recursively by $S_i^k[\mathbf{g}] = \{a_i \in A_i \mid a_i \in br_i(\pi_i) \text{ for some } \pi_i \in \Delta(S_{-i}^{k-1}[\mathbf{g}])\}$,

where we denote $S_{-i}^{k-1}[\mathbf{g}] = \prod_{j \neq i} S_i^{k-1}[\mathbf{g}]$. The set of all rationalizable actions for player $i \in \mathcal{I}$ is $S_i^\infty[\mathbf{g}] = \bigcap_{k=0}^\infty S_i^k[\mathbf{g}]$. We denote $S^\infty[\mathbf{g}] = \prod_{i \in \mathcal{I}} S_i^\infty[\mathbf{g}]$ as well as $S^k[\mathbf{g}] = \prod_{i \in \mathcal{I}} S_i^k[\mathbf{g}]$ for $k \geq 1$. We also define the set of actions that survive iterative elimination of actions that are never *strict* best response.[10] For each $i \in \mathcal{I}$, set $W_i^0[\mathbf{g}] = A_i$. Then, for $k = 1, 2, \ldots$, define $W_i^k[\mathbf{g}]$ recursively by $W_i^k[\mathbf{g}] = \{a_i \in A_i \mid \{a_i\} = br_i(\pi_i) \text{ for some } \pi_i \in \Delta(W_{-i}^{k-1}[\mathbf{g}])\}$, where we denote $W_{-i}^{k-1}[\mathbf{g}] = \prod_{j \neq i} W_i^{k-1}[\mathbf{g}]$. Finally, let $W_i^\infty[\mathbf{g}] = \bigcap_{k=0}^\infty W_i^k[\mathbf{g}]$. We denote $W^\infty[\mathbf{g}] = \prod_{i \in \mathcal{I}} W_i^\infty[\mathbf{g}]$ as well as $W^k[\mathbf{g}] = \prod_{i \in \mathcal{I}} W_i^k[\mathbf{g}]$ for $k \geq 1$. Note that $S^\infty[\mathbf{g}]$ is always nonempty, while $W^\infty[\mathbf{g}]$ may be empty (consider, e.g., games where the payoff functions are constant). But the set of normal form games $\mathbf{g}$ for which these sets coincide, $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$, is generic in the set of finite games. Our main result will be proved for this generic class of games.

We also use the following notions due to Aumann (1974) and Brandenburger and Dekel (1987). First, let us review the definition of subjective correlated equilibrium.

**Definition 2.1.** *A profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is a subjective correlated equilibrium of $\mathbf{g}$ if for all $i \in I$,*

$$\sum_{a_{-i} \in A_{-i}} \mu_i(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu_i(a_i, a_{-i}) g_i(a_i', a_{-i})$$

*for all $a_i, a_i' \in A_i$.*

As in Brandenburger and Dekel (1987), our analysis employs the refinement of subjective correlated equilibrium called a posteriori equilibrium.

**Definition 2.2.** *A profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is an $N$-subjective correlated equilibrium of $\mathbf{g}$ if it is a subjective correlated equilibrium of $\mathbf{g}$ and $\mu_i(S^N[\mathbf{g}]) = 1$ for all $i \in \mathcal{I}$.*

*A profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is an a posteriori equilibrium of $\mathbf{g}$ if it is a subjective correlated equilibrium of $\mathbf{g}$ and $\mu_i(S^\infty[\mathbf{g}]) = 1$ for all $i \in \mathcal{I}$.*

Denote by $\mathcal{E}^N[\mathbf{g}]$ the set of $N$-subjective correlated equilibria of $\mathbf{g}$ and by $\mathcal{E}[\mathbf{g}]$ the set of a posteriori equilibria of $\mathbf{g}$. Observe that $\mathcal{E}^N[\mathbf{g}]$ and $\mathcal{E}[\mathbf{g}]$ are product sets ($\mathcal{E}^N[\mathbf{g}] = \prod_{i \in \mathcal{I}} \mathcal{E}_i^N[\mathbf{g}]$ with each $\mathcal{E}_i^N[\mathbf{g}] \subset \Delta(A)$) and closed sets in $(\Delta(A))^I$.

[10] To the best of our knowledge, this notion has been first defined by Weinstein and Yildiz (2007).

We introduce a further refinement of a posteriori equilibrium.

**Definition 2.3.** *A profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is an undominated $N$-subjective correlated equilibrium of $\mathbf{g}$ if it is a $N-$subjective correlated equilibrium such that $\mu_i(W^N[\mathbf{g}]) = 1$ for all $i \in \mathcal{I}$.*

*A profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is an undominated a posteriori equilibrium of $\mathbf{g}$ if it is an a posteriori equilibrium such that $\mu_i(W^\infty[\mathbf{g}]) = 1$ for all $i \in \mathcal{I}$.*

We denote by $\mathcal{E}^{\mathrm{u}}[\mathbf{g}]$ the set of undominated a posteriori equilibrium of $\mathbf{g}$, which is again a product set. For generic games where $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$, we have $\mathcal{E}[\mathbf{g}] = \mathcal{E}^{\mathrm{u}}[\mathbf{g}]$.

Finally, we define Nash equilibrium in the following indirect way.

**Definition 2.4.** *An action distribution $\xi \in \Delta(A)$ is a correlated equilibrium of $\mathbf{g}$ if $(\mu_i)_{i \in \mathcal{I}}$ such that $\mu_i = \xi$ for all $i \in \mathcal{I}$ is a subjective correlated equilibrium of $\mathbf{g}$.*

*An action distribution $\xi \in \Delta(A)$ is a Nash equilibrium of $\mathbf{g}$ if it is a correlated equilibrium of $\mathbf{g}$ and $\xi(a) = \prod_{i \in \mathcal{I}} \xi(a_i)$ for all $a \in A$.*

## 2.2 INCOMPLETE INFORMATION PERTURBATIONS

We would like to consider incomplete information games that are close to complete information game $\mathbf{g}$.

An incomplete information game $\mathcal{U}$ consists of the set of players, $\mathcal{I}$; their action sets, $A_1, \ldots, A_I$; a countable state space, $\Omega$; a probability measure on the state space, $P_i$, for each player $i \in \mathcal{I}$; a partition of the state space, $\mathcal{Q}_i$, for each $i \in \mathcal{I}$; and a bounded payoff function, $u_i \colon A \times \Omega \to R$, for each $i \in \mathcal{I}$. The incomplete information game $\mathcal{U} = (\Omega, (P_i)_{i \in \mathcal{I}}, (\mathcal{Q}_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}})$ is said to be an incomplete information game that embeds $\mathbf{g}$. Denote by $E(\mathbf{g})$ the set of incomplete information games that embed $\mathbf{g}$. A pair $(\mathcal{U}, P_0)$ of an incomplete information game as describe above, $\mathcal{U}$, and a probability measure on the state space for the outside analyst, $P_0$, is called an incomplete information elaboration of $\mathbf{g}$.

A set of states $E \subset \Omega$ is called an event. For each $i \in \mathcal{I}$, we write $\mathcal{F}_i$ for the sigma algebra generated by $\mathcal{Q}_i$, i.e., the set of unions of events in $\mathcal{Q}_i$ together with the empty set. We say that an event $E \subset \Omega$ is *simple* if $E = \bigcap_{i \in \mathcal{I}} E_i$ where each $E_i \in \mathcal{F}_i$. We write $Q_i(\omega)$ for the element of $\mathcal{Q}_i$ containing $\omega$. We assume that $P_i(Q_i(\omega)) > 0$ for all $i \in \mathcal{I}$ and $\omega \in \Omega$. Under this assumption, the conditional probability of any $\omega'$ given $Q_i(\omega)$, $P_i(\omega' | Q_i(\omega))$, is well defined by $P_i(\omega' | Q_i(\omega)) = P_i(\omega') / P_i(Q_i(\omega))$.

We sometimes impose restrictions of possible priors.

**Definition 2.5.** *$\{P_i\}_{i \in \mathcal{I}}$ is said to have common support if $\mathrm{supp}(P_i) = \mathrm{supp}(P_j)$ for all $i, j \in \mathcal{I}$.*

By a slight abuse of language, we say that an incomplete information game $\mathcal{U}$ satisfies common support.

**Definition 2.6.** *$\mathcal{U}$ is said to satisfy the common prior assumption if $P_i = P_j$ for all $i, j \in \mathcal{I}$.*

We now define the solution concepts we use for incomplete information games. Given an incomplete information game $\mathcal{U}$, a (behavioral) strategy for player $i$ is a $\mathcal{Q}_i$-measurable function $\sigma_i \colon \Omega \to \Delta(A_i)$. Denote by $\Sigma_i$ the set of player $i$'s strategies, and let $\Sigma = \prod_{i\in\mathcal{I}} \Sigma_i$ and $\Sigma_{-i} = \prod_{j\neq i} \Sigma_j$. We write $\sigma_i(a_i|\omega)$ for the probability that action $a_i \in A_i$ is chosen at $\omega \in \Omega$ under $\sigma_i \in \Sigma_i$, and denote $\sigma_{-i}(a_{-i}|\omega) = \prod_{j\neq i} \sigma_j(a_j|\omega)$ for $\sigma_{-i} \in \Sigma_{-i}$ and $a_{-i} \in A_{-i}$ as well as $\sigma(a|\omega) = \prod_{i\in\mathcal{I}} \sigma_i(a_i|\omega)$ for $\sigma \in \Sigma$ and $a \in A$. For $\sigma \in \Sigma$ and $P_i \in \Delta(\Omega)$, we write $\sigma_{P_i} \in \Delta(A)$ for the induced action distribution with respect to $P_i$, i.e., $\sigma_{P_i}(a) = \sum_{\omega\in\Omega} P_i(\omega)\sigma(a|\omega)$ for $a \in A$.

For player $i \in \mathcal{I}$ and action $a_i \in A_i$, we write the expected payoff against a conjecture $\nu_i \in \Delta(\Omega \times A_{-i})$ as

$$U_i(a_i, \nu_i) = \sum_{\omega\in\Omega} \sum_{a_{-i}\in A_{-i}} \nu_i(\omega, a_{-i})\, u_i(a_i, a_{-i}, \omega).$$

The set of $i$'s (pure) best responses against $\nu_i \in \Delta(\Omega \times A_{-i})$ is denoted by

$$BR_i(\nu_i) = \arg\max_{a_i\in A_i} U_i(a_i, \nu_i).$$

For $i \in \mathcal{I}$ and $\sigma_{-i} \in \Sigma_{-i}$, we denote by $\sigma_{-i}^{Q_i} \in \Delta(\Omega \times A_{-i})$ the induced conjecture at $Q_i \in \mathcal{Q}_i$:

$$\sigma_{-i}^{Q_i}(\omega, a_{-i}) = P_i(\omega|Q_i)\sigma_{-i}(a_{-i}|\omega).$$

Note that $\operatorname{marg}_\Omega \sigma_{-i}^{Q_i} = P_i(\cdot|Q_i)$.

**Definition 2.7.** *A strategy profile $\sigma$ is a Bayesian Nash equilibrium of $\mathcal{U}$ if for all $i \in \mathcal{I}$,*

$$\sigma_i(a_i|\omega) > 0 \Rightarrow a_i \in BR_i\left(\sigma_{-i}^{Q_i(\omega)}\right)$$

*for all $a_i \in A_i$ and $\omega \in \Omega$.*

We also define interim correlated rationalizability. For each $i \in \mathcal{I}$, let $R_i^0[Q_i] = A_i$ for all $Q_i \in \mathcal{Q}_i$. Then, for each $i \in \mathcal{I}$, and for $Q_i \in \mathcal{Q}_i$ and for $k = 1, 2, \ldots$, define $R_i^k[Q_i]$ recursively by

$$R_i^k[Q_i] = \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \exists\, \nu_i \in \Delta(\Omega \times A_{-i}) : \\ \nu_i\big(\big\{(\omega, a_{-i}) \bigm| a_{-i} \in R_{-i}^{k-1}[\omega]\big\}\big) = 1; \\ \operatorname{marg}_\Omega \nu_i = P_i(\cdot|Q_i); \\ a_i \in BR_i(\nu_i) \end{array} \right\}$$

where we denote $R_{-i}^{k-1}[\omega] = \prod_{j\neq i} R_j^{k-1}[Q_j(\omega)]$. Let $R_i^\infty[Q_i] = \bigcap_{k=0}^\infty R_i^k[Q_i]$.

**Definition 2.8.** *A strategy $\sigma_i \in \Sigma_i$ is a rationalizable strategy of player $i$ in $\mathcal{U}$ if*

$$\sigma_i(a_i|\omega) > 0 \Rightarrow a_i \in R_i^\infty[Q_i(\omega)]$$

*for all $a_i \in A_i$ and $\omega \in \Omega$.*

This definition states that player $i$'s strategy is rationalizable if it is in the convex hull of $R_i^\infty[Q_i]$ for all $Q_i \in \mathcal{Q}_i$. While this is weaker than the standard definitions (Battigalli and Siniscalchi (2003), Dekel, Fudenberg, and Morris (2003)), our results would remain valid under any stronger notion.

Note that a Bayesian Nash equilibrium is a rationalizable strategy profile. We say that incomplete information game $\mathcal{U}$ is *dominance solvable* if $R_i^\infty[Q_i]$ is a singleton set for all $i \in I$ and $Q_i \in \mathcal{Q}_i$.

We then restate the standard definition of knowledge operator which is used in defining our main concept of robustness. Fix the information system part of an incomplete information game, $(\Omega, (P_i)_{i\in\mathcal{I}}, (\mathcal{Q}_i)_{i\in\mathcal{I}})$. The knowledge operator for player $i$, $K_i \colon 2^\Omega \to 2^\Omega$, is defined by

$$K_i(E) = \{\omega \in \Omega \mid Q_i(\omega) \subset E\}.$$

That is, $K_i(E)$ is the set of states where player $i$ knows that event $E$ is true. Let $K_*(E) = \bigcap_{i\in\mathcal{I}} K_i(E)$ be the set of states where it is mutual knowledge that event $E$ is true, i.e., where every player knows that event $E$ is true. At a state $\omega$, an event $E$ is said to be mutual knowledge at order $N$ if $\omega \in \bigcap_{n=1}^N [K_*]^n(E)$, where $[K_*]^n(\cdot)$ is defined recursively by $[K_*]^n(E) = K_*([K_*]^{n-1}(E))$. Finally, at state $\omega$, an event $E$ is said to be common knowledge if $\omega \in \bigcap_{n=1}^\infty [K_*]^n(E)$.

### 2.3 ROBUSTNESS

In this subsection, we introduce our concept of robustness of equilibria to incomplete information under non-common priors. Given an incomplete information perturbation $\mathcal{U} \in E(\mathbf{g})$, let $\Omega_{\mathbf{g}}^i$ be the set of states where the payoffs of player $i \in \mathcal{I}$ are given by $g_i$ and he knows his payoff:

$$\Omega_{\mathbf{g}}^i = \{\omega \in \Omega \mid u_i(\cdot, \omega') = g_i(\cdot) \text{ for all } \omega' \in Q_i(\omega)\}.$$

Denote $\Omega_{\mathbf{g}} = \bigcap_{i\in\mathcal{I}} \Omega_{\mathbf{g}}^i$.

**Definition 2.9.** *An incomplete information game $\mathcal{U}$ is an $(\varepsilon, N)$-perturbation of $\mathbf{g}$ if $\mathcal{U} \in E(\mathbf{g})$ and $P_i\big(\bigcap_{n=1}^N [K_*]^n(\Omega_{\mathbf{g}})\big) \geq 1 - \varepsilon$ for all $i \in \mathcal{I}$.*

*A pair $(\mathcal{U}, P_0)$ of an incomplete information game $\mathcal{U}$ and a prior distribution $P_0$ on $\Omega$ is an $(\varepsilon, N)$-elaboration of $\mathbf{g}$ if $\mathcal{U}$ is an $(\varepsilon, N)$-perturbation of $\mathbf{g}$ and $P_0\big(\bigcap_{n=1}^N [K_*]^n(\Omega_{\mathbf{g}})\big) \geq 1 - \varepsilon$.*

Observe that since $K_*(E) \subset E$ for any event $E$, we have that $[K_*]^N(\Omega_{\mathbf{g}})$ is decreasing in $N$ and thus

$\bigcap_{n=1}^{N}[K_*]^n(\Omega_{\mathbf{g}}) = [K_*]^N(\Omega_{\mathbf{g}}) \subset \Omega_{\mathbf{g}}$. Note also that if $\varepsilon' \leq \varepsilon$ and $N' \geq N$, then an $(\varepsilon', N')$-elaboration is an $(\varepsilon, N)$-elaboration.

We propose two robustness concepts, one for action distribution profiles and the other for action distributions. We underline that these robustness concepts have different interpretations.

**Definition 2.10.** *A profile of action distributions $\mu = (\mu_i)_{i\in\mathcal{I}} \in (\Delta(A))^I$ is $N$-robust to incomplete information under non-common priors, or simply, $N$-robust, if for all $\delta > 0$, there exists $\varepsilon > 0$ such that any $(\varepsilon, N)$-perturbation of* $\mathbf{g}$*, $\mathcal{U}$, has a Bayesian Nash equilibrium $\sigma$ such that $|\mu_i - \sigma_{P_i}| \leq \delta$ for all $i \in \mathcal{I}$.*

*An profile of action distributions $\mu \in (\Delta(A))^I$ is robust to incomplete information under non-common priors, or simply, robust, if there exists $N \geq 0$ such that $\mu$ is $N'$-payoff robust for all $N' \geq N$.*

Observe that if $\mu$ is $N$-robust, then it is $N'$-robust for all $N' \geq N$.

This concept is most relevant in the following situation. Imagine an analyst who considers an equilibrium of a particular complete information game. He is interested in the profile of equilibrium payoffs of this game (e.g., because of some welfare criterion he cares about). This analyst has a lack of confidence in his model. Hence, he would like to check whether the equilibrium payoff profile he considers is very sensitive to the assumption of common knowledge of payoffs. If the profile is robust in the previous sense, then ex ante (subjective) expected payoffs of each player in nearby incomplete information games will not change significantly from the complete information game situation. We do not define directly robustness for (subjective) ex ante expected payoff profiles since the ex ante payoff of each player $i$ is immediately obtained from the action distribution $\mu_i$ by $\sum_{a\in A}\mu_i(a)g_i(a)$ (whenever $\varepsilon$ is sufficiently small.).

The second concept explicitly considers the analyst's possible priors.

**Definition 2.11.** *An action distribution $\xi \in \Delta(A)$ is $N$-robust to incomplete information under non-common priors, or simply, $N$-robust, if for all $\delta > 0$, there exists $\varepsilon > 0$ such that for any $(\varepsilon, N)$-elaboration of* $\mathbf{g}$*, $(\mathcal{U}, P_0)$, $\mathcal{U}$ has a Bayesian Nash equilibrium $\sigma$ such that $|\xi - \sigma_{P_0}| \leq \delta$.*

*An action distribution $\xi \in \Delta(A)$ is robust to incomplete information under non-common priors, or simply, robust, if there exists $N \geq 0$ such that $\xi$ is $N'$-robust for all $N' \geq N$.*

This concept is relevant in a situation where the analyst is interested in ex ante expected behavior of the players, but the expectation is taken with respect to his own prior distribution, which is not necessarily equal to the priors the players may have.[11]

We will show that in generic games, a profile of action distributions $(\mu_i)_{i\in\mathcal{I}}$ (an action distribution $\xi$, resp.) of $\mathbf{g}$ is robust to incomplete information under non-common priors if and only if $(\mu_i)_{i\in\mathcal{I}}$ ($\xi$, resp.) consists of the unique rationalizable action profile of $\mathbf{g}$.[12] We want to underline that our main result will stay unchanged if we modify these robustness notions in various directions. In particular, since the nontrivial result is the "only if" part, we want to show that we can weaken this concept in many respects keeping our characterization.

**Remark 2.1.** *In the definition of robustness, we use the notion of Bayesian Nash equilibrium to be consistent with that by Kajii and Morris (1997) except for dropping the common prior assumption. One might find it questionable to use Bayesian Nash equilibrium when players do not share a common prior (Dekel, Fudenberg, and Levine (2004)). However, our results would be unchanged if we changed the solution concept to the weaker concept of interim correlated rationalizability*[13]*. Indeed, all the lemmata that are used to prove our main result are stated with rationalizable strategies.*

# 3 MAIN RESULTS

## 3.1 POINT-VALUED ROBUSTNESS

In this section, we show our main result that for any game $\mathbf{g}$ in the generic class of games where $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$, $\mathbf{g}$ has a robust equilibrium under non-common priors if and only if $\mathbf{g}$ has a unique rationalizable action profile. For $a \in A$, denote by $([a])^I$ the profile of action distributions $(\mu_i)_{i\in\mathcal{I}} \in (\Delta(A))^I$ such that $\mu_i = [a]$ for all $i \in \mathcal{I}$.

**Theorem 3.1.** *Suppose that $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$. Then, $\mu^* \in (\Delta(A))^I$ ($\xi^* \in \Delta(A)$, resp.) is robust in* $\mathbf{g}$ *if and only if $\mu^* = ([a^*])^I$ ($\xi^* = [a^*]$, resp.) for some $a^*$ such that $S^\infty[\mathbf{g}] = \{a^*\}$.*

The existence of a unique rationalizable action profile is obviously a very strong condition. For instance, the theorem does not guarantee that a unique Nash equilibrium is robust. Indeed, as proved by Kajii and Morris (1997), there exists an open set of games with a unique Nash equilibrium

[11] Kajii and Morris (1997) offer a motivating story of this type for their robustness concept under common prior, where the analyst shares a common prior with the players.

[12] The two robustness concepts a priori have no logical link and indeed are distinct if we consider their set-valued extensions, as we will see in Section 3.2. In games that have a unique rationalizable action profile, both versions of robust sets collapse to a singleton, and therefore the two point-valued concepts share the same characterization, showing their equivalence.

[13] As mentioned in the introduction, our results would be actually unchanged if we use any non-empty refinement of interim correlated rationalizability.

that is not robust.[14]

We prove the sufficiency and the necessity parts respectively in the following subsections.

#### 3.1.1 Sufficiency

In this subsection, we show the sufficiency: that if an equilibrium is a unique rationalizable action profile, then it is robust to incomplete information under non-common priors. By the finiteness of $A$, there exists $N^* \geq 0$ such that $S^n[\mathbf{g}] = S^{N^*}[\mathbf{g}]$ for all $n \geq N^*$. Recall that if action distribution profile $\mu$ (or action distribution $\xi$) is $(N-1)$-robust, then it is $N'$-robust for all $N' \geq N-1$. Hence, it suffices to show the following.

**Theorem 3.2.** *Let $N^* \geq 1$ be such that $S^n[\mathbf{g}] = S^{N^*}[\mathbf{g}]$ for all $n \geq N^*$. If $S^{N^*}[\mathbf{g}] = \{a^*\}$, then $([a^*])^I$ ($[a^*]$, resp.) is $(N^*-1)$-robust in $\mathbf{g}$.*

Thus, in order for a unique rationalizable outcome $a^*$ to be robust, mutual knowledge of order $N^*$ about the event "the payoffs are given by $\mathbf{g}$" is needed, where $N^*$ is the number of necessary elimination iteration rounds to reach the singleton $\{a^*\}$.

#### 3.1.2 Necessity

In this subsection, we show the necessity: the dominance solvability is a necessary condition for a game to have a robust equilibrium under non-common priors.

This shows that replacing common knowledge by mutual knowledge at an arbitrary high (but finite) level has far reaching consequences in particular when we drop the assumption that players share a common prior. Indeed, under the assumption of common prior, several wider classes of games have been known in which a robust equilibrium exists (see the references cited in the Introduction). The theorem below shows that all these results heavily depend on the common prior assumption.

**Theorem 3.3.** *Suppose that $W^\infty[\mathbf{g}] \neq \emptyset$. If $\mu^* \in (\Delta(A))^I$ ($\xi^* \in \Delta(A)$, resp.) is robust in $\mathbf{g}$, then $\mu^* = ([a^*])^I$ ($\xi^* = [a^*]$, resp.) for some $a^* \in A$ such that $W^\infty[\mathbf{g}] = \{a^*\}$.*

The following lemma is sufficient to prove the result. The proof of the lemma relies on a contagion argument for rationalizable action profiles or, to be more specific, for a set having the (strict) best response property (rather than for a single strict Nash equilibrium as often performed in the literature). Using this technique, in Corollary 3.5 we prove a strong result on purification of (undominated) a posteriori equilibria, which allows us to prove the necessity part.

We say that a profile of action distributions $(\mu_i)_{i\in\mathcal{I}} \in (\Delta(A))^I$ is a *strict a posteriori equilibrium* if it is an undominated a posteriori equilibrium such that for all $i \in \mathcal{I}$ and all $a_i \in A_i$ with $\mu_i(a_i) > 0$,

$$\{a_i\} = br_i(\mu_i(\cdot|a_i)).$$

**Lemma 3.4.** *Let $(\mu_i)_{i\in\mathcal{I}} \in (\Delta(A))^I$ be a strict a posteriori equilibrium such that $\mathrm{supp}(\mu_i) = \mathrm{supp}(\mu_j)$ for all $i, j \in \mathcal{I}$. Then, for any $\varepsilon > 0$ and $N \geq 0$ there exists an $(\varepsilon, N)$-perturbation of $\mathbf{g}$ such that a unique rationalizable strategy profile $\sigma$ exists and satisfies $\sigma_{P_i} = \mu_i$ for all $i \in \mathcal{I}$.*

Lemma 3.4 has the following corollary, that we can purify any undominated a posteriori equilibrium by a unique rationalizable strategy profile of a dominance solvable $(\varepsilon, N)$-perturbation.

**Corollary 3.5.** *Let $(\mu_i)_{i\in\mathcal{I}}$ be any undominated a posteriori equilibrium. For any $\delta > 0$, $\varepsilon > 0$, and $N \geq 0$, there exists an $(\varepsilon, N)$-perturbation of $\mathbf{g}$ such that a unique rationalizable strategy profile $\sigma$ exists and satisfies $|\sigma_{P_i} - \mu_i| \leq \delta$ for all $i \in \mathcal{I}$.*

Note that this corollary proves that in the generic class of games where $W^\infty[\mathbf{g}] = S^\infty[\mathbf{g}]$, any a posteriori equilibrium can be purified in the previous sense. We now prove the necessity part for our robustness result.

Let us relate our results to Weinstein and Yildiz (2007). They show that for any complete information type in the universal type space[15] (see Mertens and Zamir (1985) and Brandenburger and Dekel (1993)) and any rationalizable action profile $a^*$ of this game, there exists a dominance-solvable incomplete information game and a sequence of types drawn from this game such that (1) this sequence converges to the complete information type (with respect to the product topology in the universal type space) (2) each type of the sequence plays $a^*$. It is not difficult to show that our previous results show that the dominance-solvable incomplete information game can be an $(\varepsilon, N)$-perturbation (where $\varepsilon$ can be arbitrarily small and $N$ arbitrarily large). In addition, we show that the unique equilibrium of this $(\varepsilon, N)$-perturbation can be fully characterized using the notion of a posteriori equilibria (see Corollary 3.5). Whenever $a^*$ is a strict Nash equilibrium, the unique equilibrium of the dominance-solvable game will play $a^*$ everywhere, however, when $a^*$ is not a strict Nash equilibrium, this is not possible: $a^*$ cannot be played everywhere (as our proof reveals action profiles different from $a^*$ have also to be contagious) and the use of the contagion argument given in the proof – that relies on a posteriori equilibria – becomes crucial.

[14] While the robustness concept introduced by Kajii and Morris (1997) is different from ours, one can easily show that their example goes through if we use our formulation of robustness.

[15] Recall that a complete information type is a (degenerate) type in the universal type space where it is common knowledge that payoffs are given by the complete information game.

### 3.2 SET-VALUED ROBUSTNESS

Given that many games possess no robust equilibrium, it is natural to consider a set-valued robustness concept. Such an idea can be found for instance in Morris and Ui (2005) where the common prior is assumed. In the following, we define robustness for sets of action distribution profiles as well as those of action distributions. We give a separate treatment to these two notions since, contrary to their point-valued versions, they lead to distinct characterizations.

#### 3.2.1 Robust Sets of Action Distribution Profiles

Let us first define the robustness of sets of action distribution profiles.

**Definition 3.1.** *A product set of action distribution profiles $M = \prod_{i \in \mathcal{I}} M_i \subset (\Delta(A))^I$ is $N$-robust to incomplete information under non-common priors, or simply, $N$-robust, if it is closed, and for all $\delta > 0$, there exists $\varepsilon > 0$ such that any $(\varepsilon, N)$-perturbation of* $\mathbf{g}$ *has a Bayesian Nash equilibrium $\sigma$ such that for all $i \in \mathcal{I}$, there exists $\mu_i \in M_i$ with $|\mu_i - \sigma_{P_i}| \leq \delta$.*

*$M$ is robust to incomplete information under non-common priors, or simply, robust, if there exists $N \geq 0$ such that $M$ is $N'$-robust for all $N' \geq N$.*

Observe that if $M$ is ($N$-)robust, then any $M' \supset M$ is ($N$-)robust. In particular, $(\Delta(A))^I$ is $N$-robust for all $N \geq 0$ and thus robust. We say that $M$ is a minimal ($N$-)robust set if it is an ($N$-)robust set and no proper subset of it is an ($N$-)robust set; and that $M$ is a smallest ($N$-)robust set if it is an ($N$-)robust set and is contained in any ($N$-)robust set.

We now state and prove an existence result for minimal robust set.

**Proposition 3.6.** *Any game has a minimal $N$-robust set for each $N \geq 0$ and a minimal robust set.*

To characterize robust sets of action distribution profiles, the concept of a posteriori equilibrium is the key notion. Recall that a profile of action distributions $(\mu_i)_{i \in \mathcal{I}} \in (\Delta(A))^I$ is an a posteriori equilibrium ($N$-subjective correlated equilibrium, resp.) of $\mathbf{g}$ if it is a subjective correlated equilibrium of $\mathbf{g}$ and $\mu_i(S^\infty[\mathbf{g}]) = 1$ ($\mu_i(S^N[\mathbf{g}]) = 1$, resp.) for all $i \in \mathcal{I}$, and that $\mathcal{E}[\mathbf{g}]$ ($\mathcal{E}^N[\mathbf{g}]$, resp.) denotes the set of a posteriori equilibria ($N$-subjective correlated equilibria, resp.) of $\mathbf{g}$. We show that for generic games, a smallest robust set of action distribution profiles exists and coincides with $\mathcal{E}[\mathbf{g}]$.

**Theorem 3.7.** *Suppose that $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$. Then, $\mathcal{E}[\mathbf{g}]$ is the smallest robust set of* $\mathbf{g}$.

#### 3.2.2 Robust Sets of Action Distributions

As for the point-valued robustness concept, we consider the following alternative concept.

**Definition 3.2.** *A set of action distributions $\Xi \subset \Delta(A)$ is $N$-robust to incomplete information under non-common priors, or simply, $N$-robust, if it is closed, and for all $\delta > 0$, there exists $\varepsilon > 0$ such that for any $(\varepsilon, N)$-elaboration of* $\mathbf{g}$, *$(\mathcal{U}, P_0)$, $\mathcal{U}$ has a Bayesian Nash equilibrium $\sigma$ such that there exists $\xi \in \Xi$ with $|\xi - \sigma_{P_0}| \leq \delta$.*

*A set of action distributions $\Xi \subset \Delta(A)$ is robust to incomplete information under non-common priors, or simply, robust, if there exists $N \geq 0$ such that $\Xi$ is $N'$-robust for all $N' \geq N$.*

Observe that if $\Xi$ is ($N$-)robust, then any $\Xi' \supset \Xi$ is ($N$-)robust. In particular, $\Delta(A))$ is $N$-robust for all $N \geq 0$ and thus robust. We say that $\Xi$ is a minimal ($N$-)robust set if it is an ($N$-)robust set and no proper subset of it is an ($N$-)robust set; and that $\Xi$ is a smallest ($N$-)robust set if it is an ($N$-)robust set and is contained in any ($N$-)robust set. The existence of minimal robust set can be verified in the same way as in Proposition 3.6.

We show that for generic games, a smallest robust set of action distributions exists and coincides with the convex hull of the set of rationalizable action profiles of $\mathbf{g}$.

**Theorem 3.8.** *Suppose that $S^\infty[\mathbf{g}] = W^\infty[\mathbf{g}]$. Then, $\Delta(S^\infty[\mathbf{g}])$ is the smallest robust set of* $\mathbf{g}$.

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