Communication, consensus, and order. Who wants to speak first?

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Abstract

Parikh and Krasucki [1990] showed that if rational agents communicate the value of a function f according to a protocol upon which they have agreed beforehand, they will eventually reach a consensus about the value of f, provided a *fairness* condition on the protocol and a *convexity* condition on the function f hold. In this article, we address the issue of the influence of the protocol on the outcome of the communication process, in the case where agents communicate in order to learn information. We show that if it is common knowledge among a group of agents that some of them disagree about two protocols, then the consensus value of f must be the same according to the two protocols.

1 Introduction

Alice and Bob are sitting in front of each other, both wearing a hat that can be red or white. Suppose that the two hats are white, and that somebody asks the children what is the probability that the two hats are red. The two children observe that the other's hat is white, so they both know that the two hats cannot be red. Suppose that Alice tells Bob that the probability is zero. Bob does not learn anything about the probability, which he already knew was zero, but he understands that his own hat is white, for if it had been red, Alice would not have excluded the possibility that the two hats were red. But then if Bob tells Alice that the probability is zero, Alice will not learn anything, neither on the probability that the two hats are red, nor on the color of her hat. Indeed, since Bob now knows that his hat is white, he would have said that the probability is zero, regardless of the color of her hat. On the contrary, if Bob tells first that there are no chances for the two hats to be red, then Alice will understand that her hat is white. Therefore, if Alice wants to learn the color of her hat, she is better off speaking second. This story illustrates the following fact. From the moment that people communicate in order to be better informed, who gets to talk when is important: the communication process is not commutative, for different orders of speech may lead to different outcomes.

It is well known since Geanakoplos and Polemarchakis [1982] that in a group of rational agents, a process of simultaneous communication of posterior probabilities of an event leads to equality of all individual probabilities. Cave [1983] and Bacharach [1985] extended this result to simultaneous communication of decisions, assuming that the decision rule followed by agents satisfies a union consistency property. Yet in most economic situations where agents have to speak with each other, communication is not simultaneous. Parikh and Krasucki [1990] consider the case where agents of a group communicate the value of some function f with each other, according to a pairwise protocol upon which they have agreed beforehand. They investigate what conditions on the function f and on the communication protocol guarantee that agents eventually reach a consensus, *i.e.* that from some stage on, all the communicated values will be the same. They show that if the protocol is *fair*, that is if every participant receives information directly or indirectly from every other participant, and if the function f is *convex*, that is for all pairs of disjoint events X, X', there exists $a \in]0,1[$ such that $f(X \cup X') = af(X) + (1-a)f(X')$, then communication eventually leads to a consensus about the value of f.

In this paper, we address the question of the consequences of the choice of a communication protocol in a group of agents, in a setting where agents communicate in order to learn information from others. Different protocols may lead to different outcomes, in terms of

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consensus values of f as well as of information learned by the agents during the communication process. In particular, an agent may learn more information when she communicates with others according to some protocol α than to some protocol β. Furthermore, the most informative protocols may not be the same for all agents, in the sense that an agent may learn more information with α than with β, while another one may be better informed with β than with α. Therefore, if agents value information positively, they may disagree on the protocol they prefer to use. Depending on the state of the world, Alice and Bob may prefer to speak first or second, or may be indifferent. If neither Alice nor Bob wants to speak first, communication cannot take place. However, can we conclude that they will not learn anything from each other? In this paper, we investigate what inferences can be made by rational agents from the common knowledge that some of them disagree about the order of speech.

The following situations are both possible. First, it can be common knowledge in a group of agents that some of them prefer the same order of speech. Second, it can be common knowledge in a group of agents that some of them prefer different orders of speech. However, we show the surprising result that in the latter case, the consensus value of f must be the same whatever the order of speech. For instance, if it is common knowledge among Alice and Bob that they both prefer to speak second, then what they will communicate at the end of the day will be the same, whether Alice or Bob speaks first. We show that this result is not due to the fact that in this case, the events made common knowledge through communication are the same with the two protocols. Furthermore, we show that common knowledge that some agents disagree, or agree, on two protocols does not imply that the consensus obtained is efficient.

The paper is organized as follows. In Section 2 we describe the model and recall the basic result of Parikh and Krasucki [1990]. Section 3 defines preferences over protocols and develops the main result of this paper. In Section 4, we give some possibility results around the main Theorem, and we discuss the way we define preferences in Section 5. The main proof is given in the Appendix.

2 Preliminary notions

Let Ω be the finite set of states of the world, and 2^Ω the set of possible events. There are N agents, each agent i being endowed with a partition Π_i of Ω. When the state $\omega \in \Omega$ occurs, agent i just knows that the true state of the world belongs to $\Pi_i(\omega)$, which is the cell of i's partition that contains ω. We say that a partition Π is *finer* than a partition Π' if and only if for all ω, $\Pi(\omega) \subset \Pi'(\omega)$ and there exists ω' such that $\Pi(\omega') \subsetneq \Pi'(\omega')$. A partition Π' is coarser than a partition Π if and only if Π is finer than Π'. The partition Π_i represents the ability of agent i to distinguish between the states of the world. The coarser her partition is, the less precise her information is, in the sense that she distinguishes among fewer states of the world. As usual, we say that an agent i endowed with a partition Π_i knows the event E at state ω if and only if $\Pi_i(\omega) \subset E$. We define the meet of the partitions $\Pi_1, \Pi_2, \ldots, \Pi_N$ as the finest common coarsening of these partitions, that is the finest partition M such that for all $\omega \in \Omega$ and for all $i = 1, \ldots, N$, $\Pi_i(\omega) \subset M(\omega)$.

Common knowledge of an event E at state ω is the situation that occurs when each agent knows E at ω, each agent knows at ω that each of them knows E, each agent knows at ω that each agent knows that each agent knows... etc. Aumann [1976] showed that, given a set of N agents, the meet M of their N partitions is the partition of common knowledge among these N agents. Hence we say that an event E is common knowledge at state ω iff $M(\omega) \subset E$.

Before communicating, agents have to agree on a communication protocol that will be applied throughout the communication process, and which determines which agents are allowed to communicate at each date.

Definition 1 *A protocol α is a pair of functions (s, r) from $\mathbb{N}$ to $2^{\{1,\ldots,N\}} \times 2^{\{1,\ldots,N\}}$. If $s(t) = S$ and $r(t) = R$, then we interpret S and R as, respectively, the set of senders and the set of receivers of the communication which takes place at time t.*

We note Γ the set of protocols. Note that the type of protocols we consider are more general than that in Parikh and Krasucki [1990], for we allow for more than one agent to be senders and receivers of the communication at the same time.

Agents communicate by sending messages, which we assume to be delivered instantaneously, that is, at time t messages are simultaneously sent by every $i \in s(t)$ and heard by every $j \in r(t)$. We assume that the message sent is the *private value* of some function f defined from the set of subsets of Ω into $\mathbb{R}$. The private value of f for an agent i at state ω is $f(\Pi_i(\omega))$.

Finally, the set of states of the world Ω, the individual partitions $\Pi_1, \ldots, \Pi_N$, and the message rule f define an *information model* $I = \langle \Omega, (\Pi_i)_{1 \leqslant i \leqslant N}, f \rangle$.

We now describe how information aggregates during the communication process. At a given date t, the senders $s(t)$ selected by the protocol (s, r) send a message heard by the receivers $r(t)$. Then each individual infers the set of states of the world that are compatible with the messages possibly sent, and updates her partition accordingly. Given an information model $\langle\Omega,(\Pi_i)_{1\leqslant i\leqslant 0},f\rangle$ and a communication protocol α, we define by induction on t the set $\Pi_i^\alpha(\omega,t)$ of possible states for an agent i at time t, given that the state of the world is ω:

$\Pi_i^\alpha(\omega,0)=\Pi_i(\omega)$ and for all $t\geqslant 0$,

$\Pi_i^\alpha(\omega,t+1)=\Pi_i^\alpha(\omega,t)\cap\{\omega'\in\Omega\mid f(\Pi_j^\alpha(\omega',t))=f(\Pi_j^\alpha(\omega,t))\ \forall\, j\in s(t)\}$ if $i\in r(t)$,

$\Pi_i^\alpha(\omega,t+1)=\Pi_i^\alpha(\omega,t)$ otherwise.

Two kinds of assumptions are made on the protocol α and on the function f to guarantee that iterated communication of the value of f leads to a consensus about f. The first assumption deals with the way agents communicate. As in Parikh and Krasucki [1990], we assume that the protocol is *fair*. We adopt Koessler's [2001] adaptation of Parikh and Krasucki's definition: a protocol is *fair* if and only if every participant in this protocol communicates directly or indirectly with every other participant infinitely many times. This condition is necessary so that nobody is excluded from communication.

Assumption 1 (A1) *The protocol α is fair, that is for all pairs of players (i,j), $i\neq j$, there exists an infinite number of finite sequences $t_1\leqslant\cdots\leqslant t_K$, with $t_k\in\mathbb{N}$ for all $k\in\{1,\ldots,K\}$, such that $i\in s(t_1)$ and $j\in r(t_K)$.*

The second assumption deals with what is communicated.

Assumption 2 (A2) *f is convex, that is for all pairs of disjoint events $E,E'\subseteq\Omega$, $E\cap E'=\emptyset$, there exists $\alpha\in]0,1[$ such that $f(E\cup E')=\alpha f(E)+(1-\alpha)f(E')$.*

Note that we will have $f(E_1\cup E_2\cup\cdots\cup E_k)=\sum_{i=1}^k\alpha_i f(E_i)$, with $\alpha_i\in\,]0,1[\ \forall\ i$ and $\sum_{i=1}^k\alpha_i=1$ provided that the E_i are pairwise disjoint events. This condition is obeyed, for instance, by conditional probabilities, and implies union consistency[1] à la Cave [1983].

The next result states that, under assumptions $A1$ and $A2$, $f(\Pi_i^\alpha(\omega,t))$ has a limiting value for all ω which does not depend on i. In other words, assumptions $A1$ and $A2$ guarantee that participants in the protocol converge to a consensus about the value of f.

[1] f is union consistent if for all E,E' such that $E\cap E'=\emptyset$, $f(E)=f(E')\Rightarrow f(E\cup E')=f(E)=f(E')$.

Proposition 1 (Parikh and Krasucki (1990)) *Let $\langle\Omega,(\Pi_i)_{1\leqslant i\leqslant 0},f\rangle$ be an information model, and α a communication protocol. Under assumptions $A1$ and $A2$, there exists a date T such that for all ω, for all i,j, and all $t,t'\geq T$, $f(\Pi_i^\alpha(\omega,t))=f(\Pi_j^\alpha(\omega,t'))$.*

In the sequel, we will denote $\Pi_i^\alpha(\omega)$ the limiting value of $\Pi_i^\alpha(\omega,t)$, and Π_i^α will be called i's partition of information at consensus. $f(\Pi^\alpha(\omega))$ will denote the limiting value of $f(\Pi_i^\alpha(\omega,t))$, which does not depend on i, and will be called the consensus value of f at state ω, given protocol α.

3 Who wants to speak first? An impossibility theorem.

We assume that agents are decision-makers who prefer to be better informed in the sense of Blackwell [1983]. A partition Π is more informative in the sense of Blackwell than a partition Π' if and only if Π is a refinement of Π'. Therefore, we say that an agent is better informed with a protocol α than with a protocol β if, at the end of the day, she has a finer partition with α than with β. This induces a state-dependent preference for each agent on the set of protocols. We say that an agent prefers a protocol α to a protocol β at state ω if she thinks, at ω, that she will have a finer consensus partition with α than with β.

Definition 2 (Preferences) *Let $I:=\langle\Omega,(\Pi_i)_{1\leqslant i\leqslant 0},f\rangle$ be an information model, and α, β two distinct protocols. The event "i prefers α to β" in the model I is denoted $B_i^I(\alpha,\beta)$ and is defined as follows: $B_i^I(\alpha,\beta)=\{\omega\in\Omega\mid\forall\omega'\in\Pi_i(\omega),\ \Pi_i^\alpha(\omega')\subset\Pi_i^\beta(\omega')$ and $\exists\,\omega''\in\Pi_i(\omega)$ s.t. $\Pi_i^\alpha(\omega'')\subsetneq\Pi_i^\beta(\omega'')\}$.*

Consider again the two kids example given in the introduction. There are four states of the world, each state describing the color of Alice and Bob's hats. Formally, a state will be denoted $(A_cB_{c'})$, with $c,c'\in\{r,w\}$ denoting the colors of the two hats. We assume w.l.o.g. that Alice and Bob have a uniform prior over the set of states of the world. They communicate in turn their posterior probability that the two hats are red, namely their private value of the function $f(.)=P(\{(A_rB_r)\}\mid .)$. Each child observes the other child's hat, but does not know the color of her own, so Alice and Bob are endowed with the following partitions:[2]

$$\Pi_A=\{(A_rB_r),(A_wB_r)\}_{1/2}\ \{(A_rB_w),(A_wB_w)\}_0$$
$$\Pi_B=\{(A_rB_r),(A_rB_w)\}_{1/2}\ \{(A_wB_r),(A_wB_w)\}_0$$

[2] The subscript reflects the posterior probability in each cell.

If Alice speaks first (protocol α), individual partitions at consensus are:

$$\Pi_A^\alpha = \{(A_rB_r)\}_1\ \{(A_wB_r)\}_0\ \{(A_rB_w),(A_wB_w)\}_0$$
$$\Pi_B^\alpha = \{(A_rB_r)\}_1\ \{(A_rB_w)\}_0\ \{(A_wB_r)\}_0\ \{(A_wB_w)\}_0$$

If Bob speaks first (protocol β), individual partitions at consensus are:

$$\Pi_A^\beta = \{(A_rB_r)\}_1\ \{(A_wB_r)\}_0\ \{(A_rB_w)\}_0\ \{(A_wB_w)\}_0$$
$$\Pi_B^\beta = \{(A_rB_r)\}_1\ \{(A_rB_w)\}_0\ \{(A_wB_r),(A_wB_w)\}_0$$

If the two hats are white, *i.e.* in state (A_wB_w), Alice and Bob both end up strictly better informed when they speak second. What happens in that case? Suppose that state (A_wB_w) occurs, and that Alice and Bob stand in front of each other, waiting for the other to speak first. Alice knows that the state of the world belongs to $\{(A_rB_w),(A_wB_w)\}$. She understands that the state of the world cannot be (A_rB_w), for Bob would have been indifferent between speaking first and second at state (A_rB_w). Bob knows that the state of the world belongs to $\{(A_wB_r),(A_wB_w)\}$. He understands that the state of the world cannot be (A_wB_r), for he knows that Alice would have been indifferent between speaking first and second in that state. Therefore, knowing that the other does not want to speak first makes Alice and Bob understand that the state of the world is (A_wB_w), namely that both hats are white. From now, they have the same private information in state (A_wB_w). As they cannot learn further information from the communication process, they become indifferent between speaking first and second. This example addresses the question of whether it can be common knowledge among two persons that they disagree on their preferred order of speech. More generally, it raises the question of what inferences can be made by rational agents from the common knowledge that some of them disagree on the protocol they prefer for communicate.

We now present the main result of the paper, which states in particular that if it is the case, then the consensus value is the same according to all protocols on which the two agents disagree.

Theorem 1 *Let $I = \langle\Omega,(\Pi_i)_i, f\rangle$ be an information model such that A1 and A2 are satisfied, and α,β two protocols such that $\alpha \neq \beta$. Consider $a_1,a_2,b_1,b_2 \in \{\alpha,\beta\}$, with $a_1 \neq a_2$ and $b_1 \neq b_2$, and let us fix $i \neq j$. Assertions* (1)*,* (2)*, and* (3) *cannot be true simultaneously.*

(1) $B_i^I(a_1,a_2)$ and $B_j^I(b_1,b_2)$ are common knowledge at ω.

(2) $\omega \in B_i^I(a_1,a_2) \cap B_j^I(b_1,b_2)$ and $a_1 = b_2$.

(3) $f(\Pi^\alpha(\omega)) \neq f(\Pi^\beta(\omega))$.

Assertion (1) states that i and j's preferences about α and β are common knowledge at ω. Assertion (2) states that i and j disagree about the protocol they prefer at ω (either i prefers α and j prefers β or i prefers β and j prefers α). Assertion (3) states that the consensus value of f at state ω is not the same with the two protocols. The meaning of this theorem in the example described in the introduction is the following.

• If (1) and (2) are true, namely if it is common knowledge at some state ω that Alice and Bob prefer to speak second, then (3) is false, *i.e* the consensus value of f at ω is the same regardless of the person who speaks first.

• If (1) and (3) are true, namely if it is common knowledge at ω that Alice prefers $a_1 \in \{\alpha,\beta\}$ and Bob prefers $b_1 \in \{\alpha,\beta\}$, and if the consensus value of f differs according to whether the protocol is α or β, then (2) is false, *i.e* Alice and Bob prefer the same protocol ($a_1 = b_1$).

• If (2) and (3) are true, namely if Alice and Bob prefer different orders of speech at ω, then (1) is false, *i.e* Alice's or Bob's preferences are not common knowledge to Alice and Bob at ω.

4 Possibility results

In this section, we make precise the tightness of the Theorem 1 by giving some possibility results. All the possibility results will be proved by examples, which will be given in the most particular setting, namely when two agents publicly communicate in turn their posterior probability of some event.

Let us first emphasize that the result of the Theorem 1 is not due to the fact that two out of assertions (1), (2) and (3) cannot be true simultaneously.

Proposition 2 *(i) Assertions* (1) *and* (2) *of the Theorem 1 can be true simultaneously.*

(ii) Assertions (1) *and* (3) *of the Theorem 1 can be true simultaneously.*

(iii) Assertions (2) *and* (3) *of the Theorem 1 can be true simultaneously.*

Cast in the example given in the introduction, this proposition states that (*i*) it can be common knowledge among them that Alice and Bob prefer different orders of speech, (*ii*) Alice and Bob's preferences can be common knowledge even if the consensus probability is not the same whether Alice or Bob speaks first, and (*iii*) it is possible that Alice and Bob prefer different protocols, even if these two protocols lead to different consensus values of f.

We prove point (*i*) with the following example, which describes a situation in which it is common knowledge between Alice and Bob that both of them prefer to speak second.

Example 1 *Let $\Omega = \{1, \ldots, 19\}$ be the set of states of the world, and suppose that Alice and Bob are endowed with a uniform prior P on Ω. They communicate in turn their private value of the function $f(.) = P(\{1, 3, 6, 7, 8, 9, 13, 17, 18\} \mid .)$ and are endowed with the following partitions:*

$$\mathbf{\Pi_A} = \{1,7,8,16\}_{\frac{3}{4}}\{2,3,12,15\}_{\frac{1}{4}}\{6,11,17\}_{\frac{2}{3}} \{5,19\}_0\{4,10,14,18\}_{\frac{1}{4}}\{9,13\}_1$$

$$\mathbf{\Pi_B} = \{1,2,17\}_{\frac{2}{3}}\{3,4,6,7\}_{\frac{3}{4}}\{5,9,14,16\}_{\frac{1}{4}} \{8,12,13,18\}_{\frac{3}{4}}\{10,19\}_0\{11,15\}_0$$

If Alice speaks first (protocol α), individual partitions at consensus are:

$$\mathbf{\Pi^\alpha_A} = \{1,7,8\}_1\{16\}_0\{2,15\}_0\{3,12\}_{\frac{1}{2}}\{4,18\}_{\frac{1}{2}} \{10,14\}_0\{5,19\}_0\{6,17\}_1\{11\}_0\{9,13\}_1$$

$$\mathbf{\Pi^\alpha_B} = \{1\}_1\{2\}_0\{17\}_1\{3,4\}_{\frac{1}{2}}\{6\}_1\{7\}_1\{14\}_0\{16\}_0 \{5\}_0\{9\}_1\{8\}_1\{12,18\}_{\frac{1}{2}}\{13\}_1\{10\}_0\{19\}_0\{11\}_0\{15\}_0$$

If Bob speaks first (protocol β), individual partitions at consensus are:

$$\mathbf{\Pi^\beta_A} = \{1\}_1\{7,8\}_1\{16\}_0\{2\}_0\{3,12\}_{\frac{1}{2}}\{4,18\}_{\frac{1}{2}}\{5\}_0 \{10\}_0\{14\}_0\{15\}_0\{19\}_0\{6\}_1\{11\}_0\{17\}_1\{9\}_1\{13\}_1$$

$$\mathbf{\Pi^\beta_B} = \{1,17\}_1\{2\}_0\{3,4\}_{\frac{1}{2}}\{6,7\}_1\{5,14,16\}_0\{9\}_1 \{12,18\}_{\frac{1}{2}}\{8,13\}_1\{10,19\}_0\{11,15\}_0$$

In every state of the world, Alice and Bob prefer to speak second: $B_A(\beta,\alpha) = B_B(\alpha,\beta) = \Omega$. Therefore, it is common knowledge among Alice and Bob, in every state of the world, that Alice prefers β to α and Bob α to β. However, this does not contradict the result of Theorem 1, since for all ω, $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$. The fact that both prefer to speak second in order to be better informed is quite intuitive. When an agent is the second to speak, the first message she hears contains purely private information of the other one, whereas when she speaks first, the first message she will hear will be a join of the other's private information and her private information, so she may not learn anything. However, it can be common knowledge that two agents prefer to speak first.[3]

We prove point (*ii*) with the following example, which shows that it may be common knowledge that two agents prefer the same protocol among α and β, even if the consensus value of f is not the same whether α or β is used.

Example 2 *Let $\Omega = \{1, \ldots, 15\}$ be the set of states of the world, and suppose that Alice and Bob are endowed with a uniform prior P over Ω. They communicate in turn their private value of the function $f(.) = P(\{2, 8, 10, 12, 13, 14, 15\} \mid .)$, and are endowed with the following partitions:*

$$\mathbf{\Pi_A} = \{1,3,10\}_{\frac{1}{3}}\{2,13,15\}_1\{4,11,14\}_{\frac{1}{3}}\{5,7,12\}_{\frac{1}{3}} \{6,8,9\}_{\frac{1}{3}}$$

$$\mathbf{\Pi_B} = \{1,7,10,15\}_{\frac{1}{2}}\{2,6,8,11,13\}_{\frac{3}{5}}\{3,4,12\}_{\frac{1}{3}} \{5,9,14\}_{\frac{1}{3}}$$

If Alice speaks first (protocol α), individual partitions at consensus are:

$$\mathbf{\Pi^\alpha_A} = \{1,3,10\}_{\frac{1}{3}}\{2,13,15\}_1\{4,11,14\}_{\frac{1}{3}}\{5,7,12\}_{\frac{1}{3}} \{6,8,9\}_{\frac{1}{3}}$$

$$\mathbf{\Pi^\alpha_B} = \{1,7,10\}_{\frac{1}{3}}\{15\}_1\{2,13\}_1\{6,8,11\}_{\frac{1}{3}}\{3,4,12\}_{\frac{1}{3}} \{5,9,14\}_{\frac{1}{3}}$$

If Bob speaks first (protocol β), individual partitions at consensus are:

$$\mathbf{\Pi^\beta_A} = \{1,10\}_{\frac{1}{2}}\{3\}_0\{2,13\}_1\{15\}_1\{4,14\}_{\frac{1}{2}}\{11\}_0\{7\}_0\{9\}_0 \{5,12\}_{\frac{1}{2}}\{6,8\}_{\frac{1}{2}}$$

$$\mathbf{\Pi^\beta_B} = \{1,10\}_{\frac{1}{2}}\{7\}_0\{15\}_1\{2,13\}_1\{6,8\}_{\frac{1}{2}}\{11\}_0\{4,12\}_{\frac{1}{2}} \{3\}_0\{5,14\}_{\frac{1}{2}}\{9\}_0$$

In every state of the world, Alice and Bob prefer Bob to speak first: $B_A(\beta,\alpha) = B_B(\beta,\alpha) = \Omega$, thus it is common knowledge in every state that both prefer β to α. However, this does not contradict Theorem 1, since, for instance, $f(\Pi^\alpha(1)) \neq f(\Pi^\beta(1))$.

Finally, we prove point (*iii*) with the following example, which shows that the consensus value of f may

[3] The example is pretty tedious (288 states of the world are involved), so it is available from the authors upon request.

not be the same with α and with β in some state, even if two agents disagree on α and β in that state.

Example 3 *Let $\Omega = \{1, \ldots, 13\}$ be the set of states of the world. Suppose that Alice and Bob have a uniform prior P on Ω. They communicate in turn their private value of the function $f(.) = P(\{2, 3, 4, 8, 12\} \mid .)$, which is convex, and are endowed with the following partitions of Ω:*

$$\mathbf{\Pi_A} = \{1, 3, 7, 8\}_{\frac{1}{2}}\{2, 6, 11, 12\}_{\frac{1}{2}}\{4, 5, 10\}_{\frac{1}{3}}\{9\}_0\{13\}_0$$

$$\mathbf{\Pi_B} = \{1, 3, 5\}_{\frac{1}{3}}\{2\}_1\{4, 7, 9, 10, 12, 13\}_{\frac{1}{3}}\{6, 8\}_{\frac{1}{2}}\{11\}_0$$

If Alice speaks first (protocol α), individual partitions at consensus are:

$$\mathbf{\Pi^\alpha_A} = \{1, 3, 7, 8\}_{\frac{1}{2}}\{2\}_1\{11\}_0\{6, 12\}_{\frac{1}{2}}\{4, 10\}_{\frac{1}{2}}\{5\}_0 \{9\}_0\{13\}_0$$

$$\mathbf{\Pi^\alpha_B} = \{1, 3\}_{\frac{1}{2}}\{5\}_0\{2\}_1\{4, 10\}_{\frac{1}{2}}\{7, 12\}_{\frac{1}{2}}\{9, 13\}_0 \{6, 8\}_{\frac{1}{2}}\{11\}_0$$

If Bob speaks first (protocol β), individual partitions at consensus are:

$$\mathbf{\Pi^\beta_A} = \{1, 3, 7\}_{\frac{1}{3}}\{8\}_1\{2\}_1\{6\}_0\{11\}_0\{12\}_1\{4, 5, 10\}_{\frac{1}{3}} \{9\}_0\{13\}_0$$

$$\mathbf{\Pi^\beta_B} = \{1, 3, 5\}_{\frac{1}{3}}\{2\}_1\{4, 7, 10, \}_{\frac{1}{3}}\{12\}_1\{9, 13\}_0\{6\}_0 \{8\}_1\{11\}_0$$

The partition of common knowledge is $M = \{\Omega\}$. In state 1, Alice and Bob prefer to speak second, and $f(\Pi^\alpha(1)) \neq f(\Pi^\beta(1))$. However, this does not contradict the Theorem 1 since it is not *common knowledge* that Bob prefers to speak second. Indeed, Bob prefers to speak first in states 6 and 8.

The Theorem 1 shows in particular that common knowledge at ω that i and j disagree on α and β implies that the consensus value at ω is the same with α or β. Let us emphasize that this is not due to the fact that in this case, the events that are made common knowledge at ω by communication are the same with the two protocols. If it were the case, then the equality of consensus values of f would follow as a consequence.

Proposition 3 *Let $I = \langle \Omega, (\Pi_i)_{1 \leqslant i \leqslant 0}, f \rangle$ be an information model such that A1 and A2 are satisfied, and α, β two protocols such that $\alpha \neq \beta$. Consider $a_1, a_2, b_1, b_2 \in \{\alpha, \beta\}$, with $a_1 \neq a_2$ and $b_1 \neq b_2$, and let us fix $i \neq j$. Assertions (1), (2), and (3) can be true simultaneously.*

(1) $B^I_i(a_1, a_2)$ and $B^I_j(b_1, b_2)$ are common knowledge at ω.

(2) $\omega \in B^I_i(a_1, a_2) \cap B^I_j(b_1, b_2)$ and $a_1 = b_2$.

(3) $\Pi^\alpha(\omega) \neq \Pi^\beta(\omega)$.

We prove it with Example 1, in which it is common knowledge among Alice and Bob in every state that they both prefer to speak second, and in which the partitions of common knowledge at consensus are the following:

$$\mathbf{\Pi}^\alpha = \{1, 7, 8\}\{16\}\{2, 15\}\{3, 4, 12, 18\}\{10, 14\}\{5, 19\} \{6, 17\}\{11\}\{9, 13\}$$

$$\mathbf{\Pi}^\beta = \{1, 17\}\{6, 7, 8, 13\}\{5, 14, 16\}\{2\}\{15, 11\}\{10, 19\} \{3, 4, 12, 18\}\{9\}$$

In state 1 for instance, $\Pi^\alpha(1) = \{1, 7, 8\} \neq \Pi^\beta(1) = \{1, 17\}$, even if we have $f(\Pi^\alpha(1)) = f(\Pi^\beta(1)) = 1$ by Theorem 1.

A well-known fact in the literature[4] is that the consensus obtained through communication might well be inefficient, in the sense that it may differ from the one that would have been obtained, had agents shared their private information. Formally, given a communication protocol α, it may be the case that $f(\Pi^\alpha(\omega)) \neq f(J(\omega))$, where $J(\omega) = \bigcap_{i \in \mathcal{I}} \Pi_i(\omega)$ represents the joint information in the group. When two agents disagree on α and β, then each of them has a finer consensus partition with one of the two protocols. Similarly, when two agents agree on α and β, then both have a finer consensus partition with one of the two protocols. We may then wonder whether common knowledge that two agents agree or disagree on two protocols has positive implications in terms of efficiency of the resulting consensus.

However, Example 1 shows that common knowledge that two agents *disagree* on α and β does not imply the efficiency of the consensus obtained with α and β. Indeed, Alice and Bob prefer to speak second in every state of the world. By Theorem 1, the consensus probability is therefore the same whether Alice or Bob speaks first in every state of the world. However, the consensus probability is $1/2$ in state 3, whereas it would have been $f(\{2, 3, 12, 15\} \cap \{3, 4, 6, 7\}) = f(\{3\}) = 1$ if Alice and Bob had shared the information they privately received in state 3.

Example 2 shows that common knowledge that two agents *agree* on α and β does not imply the efficiency

[4] Since Geanakoplos and Polemarchakis [1982].

of the consensus obtained with the preferred protocol for both agents. Indeed, it is common knowledge in every state that Alice and Bob prefer protocol β. However, the consensus probability with protocol β in state 4, for instance, is $1/2$, whereas it would have been $f(\{4,11,14\} \cap \{3,4,12\}) = f(\{4\}) = 0$ if Alice and Bob had shared their private information in state 4.

5 Discussion

In this paper, we address the question of the consequences of the choice of a communication protocol in a group of agents, in a setting where individuals always prefer to learn more information from others. We show in particular that if it is common knowledge to all agents that some of them disagree on the protocol they prefer, then the consensus obtained through the communication process does not depend on the chosen communication protocol. We show that the impossibility result of the main Theorem is not due to the fact that two of the three assertions cannot be true simultaneously, and that it is enough to remove one of the three assertions to restore the possibility. Furthermore, we show that the result is not due to the fact that common knowledge that two agents disagree about two protocols implies that the common knowledge created by communication is the same with the two protocols. Finally, we show that common knowledge of an agreement on a protocol, or of a disagreement on two protocols, is not enough for the participants to reach an efficient consensus.

The way we defined preferences over protocols has two implications. First, it implies that agents prefer to be more informed *ceteris paribus*, and therefore only applies in decision settings where agents always value information positively. In game settings, agents cannot value information positively *ceteris paribus*, for they also care about the amount of information learned by their opponents during the communication process. In common interest games for instance, players may prefer protocols which lead *all* of them to be better informed. We could then define the preferences of an agent i who cares about her own information and the information possessed by an agent j as follows:

$B_i(\alpha, \beta) = \{\omega \in \Omega \mid \forall\ \omega' \in \Pi_i(\omega),\ \Pi_i^\alpha(\omega') \subset \Pi_i^\beta(\omega')$ and $\Pi_j^\alpha(\omega') \subset \Pi_j^\beta(\omega')$, and $\exists\ \omega'',\ \omega''' \in \Pi_i(\omega)$ s.t. $\Pi_i^\alpha(\omega'') \subsetneq \Pi_i^\beta(\omega'')$ and $\Pi_j^\alpha(\omega''') \subsetneq \Pi_j^\beta(\omega''')\}$

In this case, two players i and j cannot disagree about the protocol they prefer out of two protocols α and β, and no conclusion can be drawn from the common knowledge that they agree about α and β.

On the contrary, in zero-sum games a player may prefer, among two protocols, the one which leads her to be better informed, and her opponent to be less informed. The preferences of an agent i who prefers more information for herself and less information for her opponent j could be defined as follows:

$B_i(\alpha, \beta) = \{\omega \in \Omega \mid \forall\ \omega' \in \Pi_i(\omega),\ \Pi_i^\alpha(\omega') \subset \Pi_i^\beta(\omega')$ and $\Pi_j^\beta(\omega') \subset \Pi_j^\alpha(\omega')$, and $\exists\ \omega'',\ \omega''' \in \Pi_i(\omega)$ s.t. $\Pi_i^\alpha(\omega'') \subsetneq \Pi_i^\beta(\omega'')$ and $\Pi_j^\beta(\omega'') \subsetneq \Pi_j^\alpha(\omega'')\}$

In this case, i and j cannot agree on the protocol they prefer out of α and β, and common knowledge at ω of i's preferences about α and β is enough to imply that $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$.

The second implication of the way we define individual preferences over protocols is that they are not complete. Indeed, an agent may not be able to compare two protocols, since two information partitions are not always rankable in the sense of refinement. A way of completing preferences can be to define them in a more general way as follows. Let $U_i : \mathcal{D}_i \times \Omega \to \mathbb{R}$ be the utility function of agent i, where $\mathcal{D}_i$ denotes the action set of agent i. Given two protocols α and β, we say that i prefers α to β in state ω if i has a higher expected utility in ω with α than with β, *i.e.* if

$E[\max_{d \in \mathcal{D}_i} E[U_i(d,.) \mid \Pi_i^\alpha(.)] \mid \Pi_i(\omega)] >$
$E[\max_{d \in \mathcal{D}_i} E[U_i(d,.) \mid \Pi_i^\beta(.)] \mid \Pi_i(\omega)]$

However, we wouldn't have the result of Theorem 1 in this case: with such preferences, common knowledge that two agents disagree on two protocols does not imply the equality of consensus values with the two protocols, even if the two agents have the same utility function. We show it with the following example:

Example 4 *Let $\Omega = \{1,2,3,4,5,6,7\}$ be the set of states of the world, and consider three agents, Alice, Bob and Carol, endowed with a uniform probability P on Ω. Agents communicate the value of the function $f(.) = P(\{1,5\} \mid .)$ according to round-robin protocols. In protocol α, Alice whispers to Carol, who whispers to Bob, who whispers to Alice, and so on. In protocol β, Bob whispers to Carol, who whispers to Alice, who whispers to Bob, and so on. Agents are endowed with the following information partitions:*

$$\Pi_A = \{1,2,6,7\}_{\frac{1}{4}}\{3,4,5\}_{\frac{1}{3}}$$
$$\Pi_B = \{1,2,3,7\}_{\frac{1}{4}}\{4,5,6\}_{\frac{1}{3}}$$
$$\Pi_C = \{1,3,4\}_{\frac{1}{3}}\{2,5,6,7\}_{\frac{1}{4}}$$

With protocol α, individual information partitions at consensus are:

$$\Pi^\alpha_A = \{1\}_1\{2,6,7\}_0\{3,4\}_0\{5\}_1$$
$$\Pi^\alpha_B = \{1\}_1\{2,3,7\}_0\{4,6\}_0\{5\}_1$$
$$\Pi^\alpha_C = \{1\}_1\{3,4\}_0\{2,6,7\}_0\{5\}_1$$

With protocol β, individual information partitions at consensus are:

$$\Pi^\beta_A = \{1,6\}_{\frac{1}{2}}\{2,7\}_0\{3,5\}_{\frac{1}{2}}\{4\}_0$$
$$\Pi^\beta_B = \{1,3\}_{\frac{1}{2}}\{2,7\}_0\{4\}_0\{5,6\}_{\frac{1}{2}}$$
$$\Pi^\beta_C = \{1,3\}_{\frac{1}{2}}\{4\}_0\{2,7\}_0\{5,6\}_{\frac{1}{2}}$$

Let $\mathcal{D} = \{x, y\}$ be the action set of Alice and Bob, and let us assume that x is the "good" action to take in states 1, 3, and 4, and that y is the "good" action to take in states 2, 5, 6, and 7. Formally, we consider the following utility function:

$$U(x,\omega) = \begin{cases} 1 \text{ if } \omega \in \{1,3,4\} \\ 0 \text{ if } \omega \in \{2,5,6\} \end{cases}$$

$$U(y,\omega) = \begin{cases} 1 \text{ if } \omega \in \{2,5,6,7\} \\ 0 \text{ if } \omega \in \{1,3,4\} \end{cases}$$

With this utility function, Alice's decisions at the end of the communication process are the following:

$$\Pi^\alpha_A = \{1\}_x\{2,6,7\}_y\{3,4\}_x\{5\}_y$$

$$\Pi^\beta_A = \{1,6\}_{x \text{ or } y}\{2,7\}_y\{3,5\}_{x \text{ or } y}\{4\}_x$$

With protocol α, she takes the action x in states 1, 3, and 4, and the action y otherwise. Therefore, Alice always takes the "good" decision with protocol α. With protocol β however, depending of the tie-break rule, she will take action x or y in states 1, 6, 3, and 5, and will then make a mistake in cells $\{1,6\}$ and $\{3,5\}$. Therefore, Alice prefers α to β in every state of the world.

With this utility function, Bob's decisions at the end of the communication process are the following:

$$\Pi^\alpha_B = \{1\}_x\{2,3,7\}_{x \text{ or } y}\{4,6\}_{x \text{ or } y}\{5\}_y$$

$$\Pi^\beta_B = \{1,3\}_x\{2,7\}_y\{4\}_x\{5,6\}_y$$

Bob always takes the "good" decision with protocol β, whereas he necessarily makes a mistake in cells $\{2,3,7\}$ and $\{4,6\}$ with protocol α. Therefore, Bob prefers β to α in every state of the world.

As a consequence, it is common knowledge in every state of the world among Alice, Bob and Carol, that Alice and Bob have opposite preferences on α and β. However, the consensus value of f is not the same with α and β, for in state 1, for instance, the consensus value is 1 with α, and 1/2 with β.

Appendix: Proof of Theorem 1

Consider an information model $I = \langle \Omega, (\Pi_i)_i, f \rangle$, and $\alpha \neq \beta$ two protocols such that assumptions $A1$ and $A2$ are satisfied. Let us show that if points 1) and 2) of Theorem 1 are true, then point 3) is false. We show that if there exist two agents i, j and a state ω such that $B^I_i(\alpha,\beta)$ and $B^I_j(\beta,\alpha)$ are common knowledge at ω, then $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$. Clearly, the proof still holds if we invert α and β.

Recall that $M(\omega)$ denotes the meet of individual partitions before communication takes place: $M = \bigwedge_{i=1}^n \Pi_i$. We note Π^α the meet of the individual partitions at consensus, given that the protocol is α: $\Pi^\alpha = \bigwedge_{i=1}^n \Pi^\alpha_i$.

If $B^I_i(\alpha,\beta)$ and $B^I_j(\beta,\alpha)$ are common knowledge at ω, then we have

$$M(\omega) \subseteq B_i(\alpha,\beta) \cap B_j(\beta,\alpha)$$

As $\Pi^\alpha(\omega) \subseteq M(\omega)$ and $\Pi^\beta(\omega) \subseteq M(\omega)\ \forall\ \omega$, we have $\Pi^\alpha(\omega) \cap \Pi^\beta(\omega) \subseteq M(\omega)\ \forall\ \omega$. Hence we have

$$\Pi^\alpha(\omega) \cap \Pi^\beta(\omega) \subseteq B^I_i(\alpha,\beta) \cap B^I_j(\beta,\alpha) \qquad (1)$$

Consider some $\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$ (which is not empty as $\omega \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$). By definition of the meet, we have $\Pi^\alpha_i(\omega') \subseteq \Pi^\alpha(\omega')$ and $\Pi^\beta_i(\omega') \subseteq \Pi^\beta(\omega')$. As $\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$, we have $\Pi^\alpha(\omega') = \Pi^\alpha(\omega)$ and $\Pi^\beta(\omega') = \Pi^\beta(\omega)$. Then we have

$$\Pi^\alpha_i(\omega') \subseteq \Pi^\alpha(\omega) \text{ and } \Pi^\beta_i(\omega') \subseteq \Pi^\beta(\omega) \qquad (2)$$

By (1), $\omega' \in B^I_i(\alpha,\beta)$. It implies that $\Pi^\alpha_i(\omega') \subseteq \Pi^\beta_i(\omega')$. Yet $\Pi^\beta_i(\omega') \subseteq \Pi^\beta(\omega)$ by (2). Then we have

$$\Pi^\alpha_i(\omega') \subseteq \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$$

As this is true for every $\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$, we have

$$\Pi^\alpha(\omega) \cap \Pi^\beta(\omega) = \bigcup_{\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)} \Pi^\alpha_i(\omega')$$

By Proposition 1 of Parikh and Krasucki [1990], $\forall\ i, j,\ f(\Pi^\alpha_i(\omega)) = f(\Pi^\alpha_j(\omega))$ for all ω. By definition of the meet, it implies that $\forall\ \omega' \in \Pi^\alpha(\omega)$, $f(\Pi^\alpha_i(\omega')) = f(\Pi^\alpha_i(\omega))$. As f is convex, it is also union consistent, then we have $f(\Pi^\alpha(\omega) \cap \Pi^\beta(\omega)) = f(\Pi^\alpha(\omega))$.

The same reasoning applied to $\Pi^\beta_j(\omega)$ boils down to $f(\Pi^\alpha(\omega) \cap \Pi^\beta(\omega)) = f(\Pi^\beta(\omega))$.

Hence $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$ $\square$

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