Unawareness, Beliefs and Games
- Extended Abstract -

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Abstract

We define a generalized state-space model with interactive unawareness and probabilistic beliefs. Such models are desirable for many potential applications of asymmetric unawareness. We develop Bayesian games with unawareness, define equilibrium, and prove existence. We show how equilibria are extended naturally from lower to higher awareness levels and restricted from higher to lower awareness levels. We use our unawareness belief structure to show that the common prior assumption is too weak to rule out speculative trade in all states. Yet, we prove a generalized "No-trade" theorem according to which there can not be common certainty of strict preference to trade. Moreover, we show a generalization of the "No-agreeing-to-disagree" theorem.

Keywords: unawareness, awareness, type-space, Bayesian games, incomplete information, equilibrium, common prior, agreement, speculative trade, interactive epistemology.

JEL-Classifications: C70, C72, D80, D82.

1 Introduction

Unawareness is probably the most common and most important kind of ignorance. Business people invest most of their time not in updating prior beliefs, and crossing out states of the world that they previously assumed to be possible. Rather, their efforts are mostly aimed at exploring unmapped terrain, trying to figure out business opportunities that they could not even have spelled out before. More broadly, every book we read, every new acquaintance we make, expands our horizon and our language, by fusing it with the horizons of those we encounter, turning the world more intelligible and more meaningful to us than it was before (Gadamer, 1960).

With this in mind, we should not be surprised that the standard state spaces aimed at modeling knowledge or certainty are not adequate for capturing unawareness (Dekel, Lipman and Rustichini, 1998). Indeed, more elaborate models are needed (Fagin and Halpern, 1988, Modica and Rustichini, 1994, 1999, Halpern, 2001). In all of these models, the horizon of propositions the individual has in her disposition to talk about the world is always a genuine part of the description of the state of affairs.

Things become even more intricate when several players are involved. Different players may not only have different languages. On top of this, each player may also form a belief on the extent to which other players are aware of the issues that she herself has in mind. And the complexity continues further, because the player may be uncertain as to the sub-language that each other player attributes to her or to others; and so on.

Heifetz, Meier and Schipper (2006a) showed how an unawareness structure consisting of a *lattice of spaces* is adequate for modeling mutual unawareness. Every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. The join of several spaces – the lowest space at least as high as every one of them – corresponds to the fusion of the horizons of meanings expressible in these spaces.

In a companion work (Heifetz, Meier and Schipper, 2006b), we showed the precise sense in which such unawareness structures are adequate and general enough for modeling mutual unawareness. We put forward an axiom system, which extends to the multi-player case a variant of the axiom system of Modica and Rustichini (1999). We then showed how the collections of all maximally-consistent sets of formulas in our system form a canonical unawareness structure.[1] In a parallel work, Halpern and Rêgo (2005) devised another sound and complete axiomatization for our class of unawareness structures.

In this paper we extend unawareness structures so as to encompass probabilistic beliefs (Section 2) rather than only knowledge or ignorance. The definition of types (Definition 1), and the way beliefs relate across different spaces of the lattice, is a non-trivial modification of the coherence conditions for knowledge operators in unawareness structures, as formulated in Heifetz, Meier and Schipper (2006a).

With unawareness type spaces in hand, we can define Bayesian games. Here again, the definition of a strategy is not obvious. Consider a type τ with a narrow horizon, and two other types τ', τ'' with a wider horizon, that agree with the quantitative beliefs of τ regarding the aspects of reality of which τ is aware; the beliefs of τ' and τ'' differ only concerning dimensions of reality that τ does not conceive. Should the action taken by τ necessarily be some average of the actions taken by τ' and τ''? We believe that conceptually, the answer to this question is negative. When the player conceives of more parameters (e.g. motives for saving) as relevant to her decision, her optimal action (e.g. "invest in bonds" or "invest in stocks") need not be related to her optimal decision (e.g. "go shopping") when that parameter is not part of the vocabulary with which she conceives the world.[2]

The next step is to define Bayesian equilibrium. With finitely or countably many states, existence follows from standard arguments.[3] Unawareness, however, introduces a new aspect to the construction of equilibrium, namely "the tyranny of the unaware": A type who concieves of only few dimensions of reality does not have in mind types of other players with a wider horizon, so the optimal action of this type *does not depend* on the actions of these wider-horizon types. Those types, however, who assign a positive probability to this narrow-minded type, must take its action into account when optimizing.

In Section 4 we define the notion of a common prior. Conceptually, a prior of a player is a convex combination of (the beliefs of) her types (see e.g. Samet, 1998). If the priors of the different players coincide, we have a common prior. A prior of a player induces a prior on each particular space in the lattice, and if the prior is common to the players, the induced prior on each particular space is common as well.

What are the implications of the existence of a common prior? First, we extend an example from Heifetz, Meier and Schipper (2006a) and show that *speculative trade* is compatible with the existence of a common prior. This need not be surprising if one views unawareness as a particular kind of delusion, since we know that with deluded beliefs, speculative trade is possible even with a common prior (Geanakoplos, 1989). Nevertheless, we show that under a mild non-degeneracy condition, a common prior is *not compatible* with common certainty of *strict preference* to carry out speculative trade. That is, even though types with limited awareness are, in a particular sense, deluded, a common prior precludes the possibility of common certainty of the event that based on private information players are willing to engage in a zero-sum bet with *strictly positive* subjective gains to everybody. This is so because unaware types are "deluded" only concerning aspects of the world outside their vocabulary, while a common prior captures a prior agreement on the likelihood of whatever the players do have a common vocabulary. An implication of this generalized no-trade theorem is that arbitrary small transaction fees rule out speculative trade under unawareness. We complement this result be generalizing Aumann's (1976) "No-Agreeing-to-disagree" result to unawareness belief structures.

There is a growing literature on unawareness both in economics and computer science. The independent parallel work of Sadzik (2006) is closest to ours. Building on our earlier work, Heifetz, Meier and Schipper (2006a), he presents a framework of unawareness with probabilistic beliefs in which the common prior on the upmost space is a primitive. In contrast, we take types as primitives and define a prior on the entire unawareness belief structure as a convex combination of the type's beliefs. Sadzik (2006) also considers Bayesian games with unawareness, but his definition of Bayesian strategy and consequently the notion of equilibrium differs from ours. As argued above, we do not confine actions of a type with a narrow horizon to be some average of actions of the corresponding types with a wider horizon, a restriction made in Sadzik (2006). As a result, in our notion of Bayesian equilibrium every

[1] Each space in the lattice of this canonical unawareness structure consists of the maximally consistent sets of formulas in a sub-language generated by a subset of the atomic propositions.

[2] This is a crucial point in which our definition of a strategy diverges from the one in the parallel work of Sadzik (2006). This paper confines attention to a setting with a common prior, that we discuss as a special case.

[3] Recall that in standard type spaces (with no unawareness), a Bayesian equilibrium need not exist even in "non-pathological" spaces with a continuum of states (Simon 2003).

type maximizes and is certain that every other type that she is aware of maximizes as well, while in the equilibrium of actions proposed in Sadzik (2006) a type may believe that another player is irrational. Sadzik (2006) does not allow for unawareness of players, while we do (see the appendix).

A purely syntactic framework with unawareness is presented by Feinberg (2004, 2005). He applies it to games with unawareness of actions but complete information. In the appendix, we discuss an interesting example due to Feinberg (2005) and demonstrate that higher order awareness of unawareness in Feinberg (2005) corresponds to higher order belief of unawareness in our model. In a framework similar to Feinberg (2004, 2005), Čopič and Galeotti (2006) study two-player games with either unawareness of actions or unawareness of types (with a prior as a primitive). Yet, they postulate that in equilibrium beliefs over actions and payoffs must correspond to the true joint distribution over own payoffs and the opponent's actions.

Both Halpern and Rêgo (2006) and Li (2006b) present models of extensive form games with unawareness and analyze solution concepts for them.[4] Modica (2000) studies the updating of probabilities and argues that new information may change posteriors more if it implies also a higher level of awareness. A dynamic framework for a single decision maker with unawareness is introduced by Grant and Quiggin (2006). Ewerhart (2001) studies the possibility of agreement under a notion of unawareness different from the aforementioned literature. Lastly, Ahn and Ergin (2006) consider explicitly more or less fine descriptions of acts and characterize axiomatically a partition-dependent subjective expected utility representation. Since the set of all partitions of a state-space forms a complete lattice, their approach suggests a decision theoretic foundation of subjective probabilities on our lattice structure.

In the following section we present our interactive unawareness belief structure. In Section 3 Bayesian games with unawareness are developed. In Section 4 we define a common prior and investigate agreement and speculation under unawareness. Some further properties of our unawareness belief structure are relegated to the appendix, which also contains a generalization of Bayesian games in order to include unawareness of actions and players. Proofs are relegated to the appendix as well. In a separate appendix, Meier and Schipper (2007), we extend the "No-trade" theorem to infinite unawareness structures.

[4] Li (2006b) is based on her earlier work, Li (2006a).

2 Model

State Spaces: Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with the partial order $\preceq$ on $\mathcal{S}$. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β ". Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces. Each $S \in \mathcal{S}$ is a measurable space, with a σ-field $\mathcal{F}_S$.

Projections: For every S and S' such that $S' \succeq S$, there is a measurable surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Events: Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^\uparrow = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^\uparrow$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Let Σ be the set of measurable events of Ω, i.e., $D^\uparrow$ such that $D \in \mathcal{F}_S$, for some state space $S \in \mathcal{S}$.

Negation: If $(D^\uparrow, S)$ is an event where $D \subseteq S$, the negation $\neg(D^\uparrow, S)$ of $(D^\uparrow, S)$ is defined by $\neg(D^\uparrow, S) := ((S \setminus D)^\uparrow, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^\uparrow := (S \setminus D)^\uparrow$. Note that by our notational convention, we have $\neg S^\uparrow = \emptyset^S$ and $\neg \emptyset^S = S^\uparrow$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^\uparrow$ is typically a proper subset of the complement $\Omega \setminus D^\uparrow$. That is, $(S \setminus D)^\uparrow \subsetneqq \Omega \setminus D^\uparrow$.

Intuitively, there may be states in which the description of an event $D^\uparrow$ is both expressible and valid – these are the states in $D^\uparrow$; there may be states in which this description is expressible but invalid – these are the states in $\neg D^\uparrow$; and there may be states in which neither this description nor its negation are expressible – these are the states in

$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}$. Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

Conjunction and Disjunction: If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is an at most countable collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}\left(D_{\lambda}\right)\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E, S) and (F, S'), by $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^{S}$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^{S}$ we have $\emptyset^{S} \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{D_{\lambda}^{\uparrow}\right\}_{\lambda \in L}$ is defined by the de Morgan law $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \neg\left(\bigwedge_{\lambda \in L} \neg\left(D_{\lambda}^{\uparrow}\right)\right)$. Typically $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} \subsetneqq \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$, and if all D_{λ} are nonempty we have that $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$ holds if and only if all the $D_{\lambda}^{\uparrow}$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

Apart from the measurability conditions, the event-structure is analogous to Heifetz, Meier and Schipper (2006a, 2006b). See Heifetz, Meier and Schipper (2006a) for an example.

Probability Measures: Here and in what follows, we mean by events always measurable events in Σ unless otherwise stated.

Let $\Delta(S)$ be the set of probability measures on $(S, \mathcal{F}_{S})$. We consider this set itself as a measurable space endowed with the σ-field $\mathcal{F}_{\Delta(S)}$ generated by the sets $\{\mu \in \Delta(S) : \mu(D) \geq p\}$, where $D \in \mathcal{F}_{S}$ and $p \in [0, 1]$.

Marginals: For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_{S}^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_{S}.$$

Let S_{μ} be the space on which μ is a probability measure. Whenever $S_{\mu} \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_{\mu}).$$

If $S(E) \npreceq S_{\mu}$, then we say that $\mu(E)$ is undefined.

Types: I is the nonempty set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a type mapping $t_i : \Omega \rightarrow \bigcup_{\alpha \in \mathcal{A}} \Delta(S_{\alpha})$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_{S}$, for all $S \in \mathcal{S}$. We require the type mapping t_i to satisfy the following properties:*

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

Define

$$Ben_i(\omega) := \left\{\omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)\right\}.$$

For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$-based event. Note that $Ben_i(\omega)$ may not be measurable.

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for an event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection. Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r_{S_{\beta}}^{S_{\alpha}}\right)_{S_{\beta} \preceq S_{\alpha}}, (t_i)_{i \in I} \right\rangle$ an interactive unawareness belief structure.*

Awareness and Unawareness: The definition of awareness is analogous to the definition in unawareness knowledge structures (see Remark 6 in Heifetz, Meier and Schipper, 2006b).

Definition 3 *For $i \in I$ and an event E, define the awareness operator*

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S), S \succeq S(E)\}$$

if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by $A_i(E) := \emptyset^{S(E)}$ otherwise.

Proposition 1 *If E is an event, then $A_i(E)$ is an $S(E)$-based event.*

Definition 4 *The unawareness operator of individual $i \in I$ on events is now defined by*

$$U_i(E) = \neg A_i(E).$$

Note that the definition of our negation and Proposition 1 imply that if E is an event, then $U_i(E)$ is an $S(E)$-based event.

Note further that Definition 3 and 4 apply also to events that are not necessarily measurable.

Belief: The p-belief-operator is defined as usual (see for instance Monderer and Samet, 1989):

Definition 5 *For $i \in I$, $p \in [0,1]$ and an event E, the p-belief operator is defined, as usual, by*

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by $B_i^p(E) := \emptyset^{S(E)}$ otherwise.

Proposition 2 *If E is an event then $B_i^p(E)$ is an $S(E)$-based event.*

The p-belief operator has the standard properties. Moreover, all properties of unawareness that have been proposed in the literature obtain (see Heifetz, Meier and Schipper, 2007, for details).

3 Bayesian Games with Unawareness

For simplicity, we consider first Bayesian games with unawareness in which every player is aware of all players and of all her and other's actions. In Heifetz, Meier and Schipper (2007), we generalize our theory to allow also for unawareness of actions and players. For notational convenience, we restrict ourselves in this section to a finite set of players, finite sets of actions, finite state-spaces, and assume that for each $S \in \mathcal{S}$, $\mathcal{F}_S = 2^S$.

Definition 6 *A Bayesian game with unawareness of events consists of an unawareness belief structure $\underline{\Omega} = \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ that is augmented by a tuple $\left\langle (M_i)_{i \in I}, (u_i)_{i \in I} \right\rangle$ defined as follows: For each player $i \in I$, there is*

(i) a finite nonempty set of actions M_i, and

(ii) a utility function $u_i : \left(\prod_{i \in I} M_i\right) \times \Omega \longrightarrow \mathbb{R}$.

The interpretation is as follows: At the beginning of a game, a state $\omega \in \Omega$ is realized. Player i does not observe the state but receives a signal $t_i(\omega)$ that provides her with some information about the state or projections thereof to lower spaces. I.e., if ω obtains, player i is of type $t_i(\omega)$. This signal is a belief about the likelihood of events on a certain space. A player's utility depends on her action, the actions chosen by other players as well as the state. Since players may be uncertain about the state ω, we assume below that the player's preference is represented by the expected value of the utility function on action-profiles of players and states, where the expectation is taken with respect to the player i's type $t_i(\omega)$ and the types' mixed strategies. This game allows for unawareness of possibly payoff relevant events.

Let $\Delta(M_i)$ be the set of mixed strategies for player $i \in I$, that is, the set of probability distributions on the finite set M_i.

Definition 7 *A strategy of player i in a Bayesian game with unawareness of events and actions is a function $\sigma_i : \Omega \longrightarrow \Delta(M_i)$ such that for all $\omega \in \Omega$,*

(i) $\sigma_i(\omega) \in \Delta(M_i)$,

(ii) $t_i(\omega) = t_i(\omega')$ implies $\sigma_i(\omega) = \sigma_i(\omega')$.

In Bayesian games with unawareness we interpret a Bayesian strategy from an interim point of view: Given a player i and type $t_i(\omega)$, she has an "awareness level" $S_{t_i(\omega)} \in \mathcal{S}$. That is, she can consider strategies of her opponents in $l(S_{t_i(\omega)})$, where $l(S) := \{S' \in \mathcal{S} : S' \preceq S\}$ is the complete sublattice of $\mathcal{S}$ with S being the upmost space. This interpretation is sound precisely because of Propositions 4 and 5 below: To best-respond to the strategies of the other player-types, a type of a player needs only to reason about the strategies of player-types that she is aware of. Only strategies of these player-types enter in her utility maximization problem.

Denote $\sigma_{S_{t_i(\omega)}} := \left((\sigma_j(\omega'))_{j \in I}\right)_{\omega' \in S_{t_i(\omega)}}$. A component $\sigma_j(\omega')$ of the strategy profile $\sigma_{S_{t_i(\omega)}}$ is the strategy of the *player-type* $(j, t_j(\omega'))$. $\sigma_{S_{t_i(\omega)}}$ is the profile of all player-types' strategies.

The *expected utility* of player-type $(i, t_i(\omega))$ from the strategy profile $\sigma_{S_{t_i(\omega)}}$ is given by

$$\begin{aligned} &U_{(i,t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) \\ &\quad := \int_{\omega' \in S_{t_i(\omega)}} \sum_{m \in \prod_{j \in I} M_j} \prod_{j \in I} \sigma_j(\omega')(m_j) \\ &\quad \cdot u_i(m, \omega')\, dt_i(\omega)(\omega'). \end{aligned} \tag{1}$$

$\sigma_j(\omega')(m_j)$ is the probability with which the player-type $(j, t_j(\omega'))$ plays the action $m_j \in M_j$. $\prod_{j \in I} \sigma_j(\omega')(m_j)$ is the joint probability with which the action profile $m = (m_j)_{j \in I}$ is played by the players. This action profile gives the utility $u_i((m_j)_{j \in I}, \omega')$ to player i in state ω'. The term $\sum_{m \in \prod_{j \in I} M_j} \prod_{j \in I} \sigma_j(\omega')(m_j) \cdot u_i\left((m_j)_{j \in I}, \omega'\right)$ is player i's expected utility from the strategy profile $(\sigma_j(\omega'))_{j \in I}$ at the state ω'. However, at a state ω, the player, in general, does not know the state, but only his type $t_i(\omega)$, and so he evaluates his utility with the expectation with respect to the probability measure $t_i(\omega)$.

Definition 8 (Equilibrium) *An* equilibrium *of a Bayesian game with unawareness of events* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *is a Nash equilibrium of the strategic game defined by:*

(i) $\{(i, t_i(\omega)) : \omega \in \Omega$ *and* $i \in I\}$ *is the set of players,*

and for each player $(i, t_i(\omega))$,

(ii) the set of mixed strategies is $\Delta(M_i)$, *and*

(iii) the utility function is given by equation (1).

Proposition 3 (Existence) *Let* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *be a Bayesian game with unawareness of events. If* I, Ω, *and* $(M_i)_{i \in I}$ *are finite, then there exists an equilibrium.*

Note that contrary to an ordinary Bayesian game, the game is not "common knowledge" among the players. Let $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ be a Bayesian game with unawareness of events. At $\omega \in \Omega$, the game conceived by player i is $\left\langle l(S_{t_i(\omega)}), \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$, where the lattice of spaces is replaced with the sublattice $l(S_{t_i(\omega)})$ with $S_{t_i(\omega)}$ as the upmost space, $\left(r^{S_\alpha}_{S_\beta}\right)$ are restricted to $S_\alpha, S_\beta \in l(S_{t_i(\omega)})$, and accordingly, the domains of the t_i and u_i are restricted to $\bigcup_{S \in l(S_{t_i(\omega)})} S$. Type $t_i(\omega)$ of player i can conceive of all events expressible in the spaces of the sublattice $l(S_{t_i(\omega)})$. For $S \in \mathcal{S}$, we call $\left\langle l(S), \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ the *S-partial Bayesian game with unawareness of events.*

Proposition 4 ("Upwards Induction") *Given a Bayesian game with unawareness of events* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$, *define for* $S', S'' \in \mathcal{S}$ *with* $S' \preceq S''$ *the* S'*-partial (resp.* S''*-partial) Bayesian game with unawareness of events. If* I, Ω, *and* $(M_i)_{i \in I}$ *are finite, then for every equilibrium of the* S'*-partial Bayesian game, there is an equilibrium of the* S''*-partial Bayesian game in which equilibrium strategies of player-types in* $\{(i, t_i(\omega)) : \omega \in \bigcup_{S \in l(S')} S$ *and* $i \in I\}$ *are identical with the equilibrium strategies in the* S'*-partial Bayesian game.*

This proposition suggests a procedure for constructing equilibria in Bayesian games with unawareness. We start with an equilibrium in the $\hat{S}$-partial Bayesian game with unawareness, where $\hat{S}$ denotes the greatest lower bound space (the meet) of the lattice, and extend it step-by-step to higher spaces by finding a fixed-point taking the strategies of player-types in the respective lower spaces as given.

For some strategic situations, Proposition 4 suggests that players which are unaware may have commitment power compared to players with a "higher awareness level". This is so because types with "lower awareness levels" do not react to types of which they are unaware. Types with "higher awareness" must take strategies of types with "lower awareness" as given. In some strategic situations, the value of awareness may be negative.

Proposition 5 *Let* $\left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i), (u_i) \right\rangle$ *be a Bayesian game with unawareness of events. Define for* $S', S'' \in \mathcal{S}$ *with* $S' \preceq S''$ *the* S'*-partial (resp.* S''*-partial) Bayesian game with unawareness of events. Then for every equilibrium of the* S''*-partial Bayesian game there is an equilibrium of the* S'*-partial Bayesian game in which the equilibrium strategies of player-types in* $\{(i, t_i(\omega)) : \omega \in \bigcup_{S \in l(S')} S$ *and* $i \in I\}$ *are identical with the equilibrium strategies of the* S''*-partial Bayesian game.*

We conclude this section with a simple example that touches a prime theme of unawareness: novelties, inventions and innovations.

Example 1 (The Mathematician's Dilemma) Two brilliant mathematicians, Emmy and Nicolai,

Figure 1: Information Structure and Payoffs in the Mathematician's Dilemma

ω_1

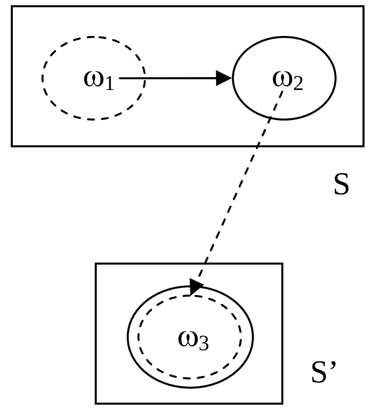
Emmy \ Nicolai	now	later
now	490, 490	1080, 0
later	0, 1080	590, 590

ω_2

Emmy \ Nicolai	now	later
now	990, -10	1080, 0
later	1000, 80	1090, 90

ω_3

Emmy \ Nicolai	now	later
now	-10, -10	80, 0
later	0, 80	90, 90



consider to compete on solving a problem in mathematics. Solving the problem now rather than later is costly because there are also other unsolved problems they could try to solve. We assume that the costs of solving it now rather than later are 100K dollars for either player. Moreover, we assume that a solution to this problem is prized at 180K dollars. This is to be shared if both solve it at the same time. If only one solves it now, and the other later, then the latter gets nothing and the former 180K.

When solving the problem, any of the two mathematicians could be quite unexpectedly aware of a brilliant idea that would not only solve their problem but also prove the Riemann Hypothesis. This chance-discovery is not foreseen by anybody in the profession. Luckily the Clay Mathematics Institute of Cambridge, M.A., offers a reward of 1 million dollars for the proof of the Riemann Hypothesis. We assume that this prize is shared if both provide a proof at the same time. If one is first, then he gets the entire prize.

To model their awareness and beliefs, we consider two state-spaces S and S'. We assume that S is richer than S' in the sense that whenever a player believes some state in S, then (s)he is aware of the brilliant idea. A player's belief at each state is given by a probability distribution on one of those spaces. To be precise, consider the information structure in Figure 1. There are three states, ω_1 and ω_2 in S and ω_3 in S'. The solid (resp. dashed) lines/ovals belong to Emmy (resp. Nicolai).

At each state, both mathematicians have two actions: work on it now or later. In Figure 1, we also depict the payoff matrix whose entries correspond to the story above.

The payoff matrices together with the information structure constitute a Bayesian game with unawareness. What could be a solution? An equilibrium should specify for each state an optimal strategy profile given the beliefs and awareness of the players at that state. We start by considering optimal strategies at $\omega_3 \in S'$, the space where both players are unaware of the brilliant idea. In the symmetric game at ω_3, (later) is the dominant action for both players.

Next consider the state $\omega_2 \in S$. At this state, Emmy is aware of the brilliant idea because she believes $\omega_2 \in S$ with probability one. In contrast, Nicolai is unaware of it because at ω_2 he believes in $\omega_3 \in S'$ with probability one. Note that at ω_2 Emmy believes that Nicolai is unaware of it. Both player's dominant action is (later).

Finally, consider the state $\omega_1 \in S$. At this state both Emmy and Nicolai are aware of the brilliant idea since their "information sets" lie in S. But since Emmy's "information set" at ω_1 is $\{\omega_2\}$ and at ω_2 Nicolai's "information set " is $\{\omega_3\} \subset S'$, Emmy believes that Nicolai is unaware of it. Moreover, Nicolai believes that Emmy is aware of it, and he believes that Emmy believes that he (Nicolai) is unaware of it. So for Emmy the dominant action at ω_1 is later but for Nicolai it is now. That is, even though Emmy is aware of the brilliant idea and could solve the Riemann Hypothesis, she won't receive the desired award.

Note that the result of the example continues to hold if beliefs are slightly perturbed. E.g., at ω_1 and ω_2 Emmy could assign probability $\frac{1}{1000}$ to ω_1 and $\frac{999}{1000}$ to ω_2. □

4 Common Prior, Agreement, and Speculation

Common Belief: We define mutual and common belief as usual (e.g. Monderer and Samet, 1989):

From now on, we assume that the set of individuals I

is at most countable.

Definition 9 *The mutual p-belief operator on events is defined by*

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n (E).$$

If E is a measurable event, then $B^p(E)$ and $CB^1(E)$ are $S(E)$-based events.

We say that an event E is *common certainty* at $\omega \in \Omega$ if $\omega \in CB^1(E)$.

Priors and Common Priors:

Definition 10 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*
2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

$$P_i^S\left(E \cap S \cap A_i(E)\right) = \int_{S \cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{2}$$

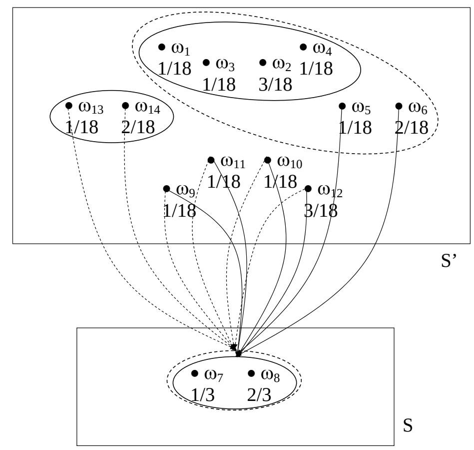
$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a common prior if P is a prior for every player $i \in I$.

In particular, if S is finite or countable, equality (2) holds if and only if

$$P_i^S\left(E \cap S \cap A_i(E)\right) = \sum_{s \in S \cap A_i(E)} t_i(s)(E)\, P_i^S(\{s\}). \tag{3}$$

Figure 2 illustrates a common prior in an unawareness belief structure. Odd (resp. even) states in the upper space project to the odd (resp. even) state in the lower space. There are two individuals, one indicated by the solid lines and ellipses and another by intermitted lines and ellipses. Note that the ratio of probabilities over odd and even states in each "information cell" coincides with the ratio in the "information cell" in the lower space.

Figure 2: Illustration of a Common Prior



Speculation: With a probabilistic version of the speculative trade example in Heifetz, Meier and Schipper (2006a), one can show that a common prior is not sufficient to rule out speculation under unawareness. In particular, we can have common certainty of willingness to trade but each party strictly prefers to trade (see Heifetz, Meier and Schipper, 2007, for details). Despite this counter example to the "No-trade" theorems, we can prove below a generalized version according to which, if there is a common prior, then there can not be common certainty of strict preference to trade. That is, even with unawareness not "everything goes". We find this surprising, because unawareness can be interpreted as a special form of delusion: At a given state, a player may be certain of states in a very different lower state-space. It is know that speculative trade is possible in delusional standard state-space structures with a common prior.

Definition 11 *A common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is* non-degenerate *if and only if for all $i \in I$ and $\omega \in \Omega$: If $t_i(\omega) \in \triangle(S')$, for some S', then $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ and $P^S\left(\left([t_i(\omega)] \cap S'\right)^{\uparrow} \cap S\right) > 0$ for all $S \succeq S'$.*

Definition 12 *Let x_1 and x_2 be real numbers and v a random variable on Ω. Define the sets $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d\left(t_1(\omega)\right)(\cdot) \leq x_1\right\}$ and $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d\left(t_2(\omega)\right)(\cdot) \geq x_2\right\}$. We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_1^{\geq x_2}$).*

Note that the sets $E_1^{\leq x_1}$ or $E_2^{\geq x_2}$ may not be events in our unawareness-belief structure, because $v(\omega) \neq$

$v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p-belief, mutual p-belief and common certainty for measurable subsets of Ω, and show that standard properties obtain as well (see Meier and Schipper, 2007).

Theorem 1 *Let Ω be a finite unawareness belief structure and $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ be a non-degenerate common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a random variable $v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.*

Together with our example of speculative trade under unawareness we conclude that a common prior does not rule out speculation under unawareness but it can never be common certainty that both players expect to strictly gain from speculation. The theorem implies immediately as a corollary that given a non-degenerate common prior, *arbitrary small transaction costs rule out speculative trade under unawareness.*

In Meier and Schipper (2007), we extend above "No-trade" theorem to infinite unawareness-belief structures. To this extent we introduce a topological unawareness belief structure and consider as a technical device a "flatted" structure with the union of all spaces in the lattice as the state-space.

Agreement: For an event E and $p \in [0,1]$ define the set $[t_i(E) = p] := \{\omega \in \Omega : t_i(\omega)(E) = p\}$, if $\{\omega \in \Omega : t_i(\omega)(E) = p\}$ is nonempty, and otherwise set $[t_i(E) = p] := \emptyset^{S(E)}$. Note that $[t_i(E) = p]$ is a $S(E)$-based event.

The following proposition is a generalization of the standard *"No-Agreeing-to-Disagree" theorem* (Aumann, 1976):

Proposition 6 *Let G be an event and $p_i \in [0,1]$, for $i \in I$. Suppose there exists a common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that for some space $S \succeq S(G)$ we have $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$. Then $p_i = p_j$, for all $i, j \in I$.*

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