A Canonical Model for Interactive Unawareness - Extended Abstract -

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Abstract

Heifetz, Meier and Schipper (2006) introduced a generalized state-space model that allows for non-trivial unawareness among several individuals and strong properties of knowledge. We show that this generalized state-space model arises naturally if states consist of maximally consistent sets of formulas in an appropriate logical formulation.

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1 Introduction

Unawareness refers to lack of conception rather than to lack of information. In standard models of incomplete information, decision makers share the conception of the interactive decision problem but may have asymmetric information. While models of asymmetric information are common in game theory and economics, the modelling of unawareness in interactive decision making proves to be a tricky task. Modica and Rustichini (1994) and Dekel, Lipman and Rustichini (1998) showed that standard state-space models of asymmetric information preclude non-trivial forms of unawareness.

Heifetz, Meier and Schipper (2006) introduced a generalized state-space model that allows for non-trivial unawareness among several individuals and strong properties of knowledge. Such an *unawareness structure* involves as primitives a lattice of state-spaces ordered according to their strength of "expressive power," and suitable projections among them. The possibility set of one individual in a state of one space may reside in a lower space, while the possibility set of a second individual in one of these possible states may reside in yet a lower space. The former phenomenon reflects the fact that the first individual is unaware of some aspects of reality, and conceives fewer dimensions of the situation than there actually are; the latter reflects the fact that the first individual considers as possible that the second individual is unaware of some aspects of which the first individual *is* aware. That's how the model captures *mutual* beliefs about unawareness.

In this paper we substantiate this construction with logical building blocks. Such an investigation of foundations is indeed necessary. Any structure for modelling asymmetric cognition – even if it has the desired properties – begs the question whether the model *itself* should not be an object for further uncertainties of the individuals. One way to show that a model *is* comprehensive is to describe in minute detail the beliefs and mutual beliefs of all individuals in each state. If each relevant combination of such beliefs are described in some state of the model, then we are convinced that the phenomena we aim at modelling is captured by our construction.

For the case of knowledge and mutual knowledge, such a detailed substantiation of the beliefs in the standard partition model has been carried out by constructing the canonical model of all maximally-consistent sets of formulas in the S5 system of epistemic logic (Aumann, 1999). This is the standard propositional logic (with negations, conjunctions and disjunctions of formulas), augmented with a knowledge modality for each individual: For each formula φ and each individual i, there is also a formula $k_i\varphi$ ("individual i knows φ"). The axioms of the system then specify (on top of the standard logical tautologies) that what an individual knows is true, and that the individual knows what she knows and what she ignores. The canonical construction in which the states are the maximally-consistent sets of formulas in this logical system turns out to have a natural partition structure, and the knowledge of events in this partition model reflects exactly the knowledge of formulas in the states. That's how the canonical model is substantiated by the internal structure of its own states.

In this paper we aim at an analogous foundation for the case of unawareness. Two main differences arise.

First, the axiom system has to be amended: An individual knows that she doesn't know some fact only if she is aware of that fact. Our axiom system manifests this feature, and is thus (equivalent to) a multi-person version of axiom systems proposed by Modica and Rustichini (1999) and Halpern (2001).

Second, in the lattice of spaces of an unawareness structure, only the states in the upper-most space are full descriptions of all aspects of reality relevant to the interaction among the individuals. States in lower spaces are subjective portraits of situations, in the mind of individuals who are unaware of some of these relevant dimensions. Therefore, to construct these subjective descriptions, one has to use a sub-language, with a proper subset of the primitive propositions of the logical syntax.

With these modifications done, we are able to accomplish our mission: The collections of maximally-consistent sets of formulas (across all sub-languages) constitute an unawareness structure (Theorem 1), in which knowledge of events reflects exactly the knowledge of formulas in states (Theorem 2).

This result entails an important corollary. Logicians are often interested whether an axiom system is sound and complete with respect to a given family of models. Soundness means that every theorem derivable from the axioms obtains in all states of all these models. Completeness means the reverse implication: Every formula that obtains in all states of all the models in the family is provable from the axioms of the system. Our results imply that the axiom system we devised is sound and complete with respect to the family of unawareness structures (Theorem 3).[1]

The caveat here is that each space in the lattice of an unawareness structure is associated with a sub-language (determined by a subset of the primitive propositions), and that higher spaces in the lattice are associated with richer sub-languages. In a state which is a subjective description of reality in the mind of an individual who is unaware of a primitive proposition φ, neither φ nor $\neg\varphi$ would obtain. If one were to insist on a truth value for φ in such a state, it would have to be "mu" – a third value, distinct than "true" and "false". This would lead to a three-valued logic, with an extended axiom system that accounts for the third truth value. In parallel work, Halpern and Rêgo (2005) have developed such a three-valued sound and complete axiomatization for the family of our unawareness structures.

[1] In fact, even *strong* soundness and completeness are implied: A set of formulas Γ is consistent (that is, free of contradiction) if and only if there is a state in a model in which Γ obtains.

2 Syntax and Axioms

Let X be the nonempty set of atomic formulas or primitive propositions, and let I be the set of individuals. We assume I to be nonempty, but otherwise no restriction is imposed (so I could be infinite, even uncountable).

The syntax is the usual multi-person modal syntax with knowledge modalities k_i, "negation" $\neg$, "and" $\wedge$ and a constant $\top$ for truth. As usual, $\vee$, $\rightarrow$, and $\longleftrightarrow$ are abbreviations, defined in the usual way.

The *set of formulas* $\mathcal{L}_X$ is the smallest set such that:

- $\top$ is a formula,
- every $x \in X$ is a formula,
- if φ is a formula, then $\neg\varphi$ is a formula,
- if φ and ψ are formulas, then $\varphi \wedge \psi$ is a formula,
- if φ is a formula, then $k_i\varphi$ is a formula.

The *awareness* modality is defined by $a_i\varphi := k_i\varphi \vee k_i\neg k_i\varphi$ and the *unawareness* modality is defined by $u_i\varphi := \neg a_i\varphi$.

For a formula φ of our language, define $\Pr(\varphi)$ to be the set of primitive propositions occurring in φ.

The axiom system $S5$ (see for example Chellas, 1980, pp. 14 or Fagin et al. 1995, p. 56) corresponds to standard partitional models usually applied in economics. This axiom system is weakened to the following system:

- All substitution instances of valid formulas of *Propositional Calculus* including the formula $\top$, (PC)
- the inference rule *Modus Ponens*:

$$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi}, \tag{MP}$$

- the *Axiom of Truth*:

$$k_i\varphi \rightarrow \varphi, \tag{T}$$

- the *Axiom of Positive Introspection*:

$$k_i\varphi \to k_ik_i\varphi. \tag{4}$$

- the *Propositional Awareness axioms*:

$$\begin{array}{lrcl} 1. & a_i\varphi & \longleftrightarrow & a_i\neg\varphi, \\ 2. & a_i\varphi \wedge a_i\psi & \longleftrightarrow & a_i(\varphi \wedge \psi), \\ 3. & a_i\varphi & \longleftrightarrow & a_ik_j\varphi, \text{for } j \in I. \end{array} \tag{PA}$$

- and the inference rule *RK-Inference*: For all natural numbers $n \geq 1$: If φ_1, φ_2, ..., φ_n and φ are formulas such that $\Pr(\varphi) \subseteq \bigcup_{i=1}^{n} \Pr(\varphi_i)$ then

$$\frac{\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n \to \varphi}{k_i\varphi_1 \wedge k_i\varphi_2 \wedge ... \wedge k_i\varphi_n \to k_i\varphi}. \tag{RK}$$

A *tautology* is a valid formula of Propositional Calculus.

The set of *theorems* is the smallest set of formulas that contain all the axioms (that is all the substitution instances of valid formulas of Propositional Calculus, Truth, the Propositional Awareness Axioms and Axiom (4)) and that is closed under the inference rules Modus Ponens and RK-Inference.

Let Γ be a set of formulas and φ a formula. A *proof of* φ *from* Γ is a finite sequence of formulas such that the last formula is φ and such that each formula is a formula in Γ, a theorem of the system or inferred from the previous formulas by Modus Ponens. If there is a proof of φ from Γ, then we write $\Gamma \vdash \varphi$. In particular, $\vdash \varphi$ means that φ is a theorem. If $\Gamma \vdash \varphi$, we say that Γ *implies* φ *syntactically*.

A set Γ of formulas is *consistent* if and only if there is no formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. A set Γ of formulas is *inconsistent*, if it is not consistent.

RK-inference implies immediately that $k_i\varphi \wedge k_i(\varphi \to \psi) \to k_i\psi$ and $k_i\varphi \wedge k_i\psi \to k_i(\varphi \wedge \psi)$ are theorems.

The axiomatization implies that $a_i\varphi \to k_ia_i\varphi$ is a theorem.

For every formula φ we have that $a_i\varphi \leftrightarrow \bigwedge_{x \in \Pr(\varphi)} a_ix$ is a theorem. Hence an individual i is aware of a formula if and only if she is aware of all primitive propositions in this formula.

If φ is a theorem, then $a_i\varphi \to k_i\varphi$ is theorem. Hence $\{a_i\varphi\} \vdash k_i\varphi$, if φ is a theorem.

That is, the following weaker form of the standard inference rule Necessitation obtains: Whenever an individual is aware of a theorem, then he knows that theorem.

3 Unawareness Structures

In this section we recall the definition of unawareness structures in Heifetz, Meier and Schipper (2006).

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state spaces*, with a partial order $\preceq$ on $\mathcal{S}$. Denote by $\Sigma = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces.

For every S and S' such that $S' \succeq S$, there is a surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $s \in S'$, denote $s_S = r_S^{S'}(s)$. If $B \subseteq S'$, denote $B_S = \{s_S : s \in B\}$.

Denote $g(S) = \{S' : S' \succeq S\}$. For $B \subseteq S$, denote $B^\uparrow = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(B)$.

An *event* is a pair (E, S), where $E = B^\uparrow$ with $B \subseteq S$, where $S \in \mathcal{S}$. B is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Σ is an event.

If $(B^\uparrow, S)$ is an event where $B \subseteq S$, the negation $\neg(B^\uparrow, S)$ of $(B^\uparrow, S)$ is defined by $\neg(B^\uparrow, S) := ((S \setminus B)^\uparrow, S)$. Abusing notation, we write $\neg B^\uparrow := (S \setminus B)^\uparrow$. Note that by our notational convention, we have$\neg S^\uparrow = \emptyset^S$ and $\neg\emptyset^S = S^\uparrow$, for each space $S \in \mathcal{S}$. $\neg B^\uparrow$ is typically a proper subset of the complement $\Sigma \setminus B^\uparrow$. That is, $(S \setminus B)^\uparrow \subsetneqq \Sigma \setminus B^\uparrow$. Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

If $\left\{\left(B_\lambda^\uparrow, S_\lambda\right)\right\}_{\lambda \in L}$ is a set of events (with $B_\lambda \subseteq S_\lambda$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(B_\lambda^\uparrow, S_\lambda\right)$ is defined by $\bigwedge_{\lambda \in L}\left(B_\lambda^\uparrow, S_\lambda\right) := \left(\left(\bigcap_{\lambda \in L} B_\lambda^\uparrow\right), \sup_{\lambda \in L} S_\lambda\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_\lambda$ exists. If $S = \sup_{\lambda \in L} S_\lambda$, then we have $\left(\bigcap_{\lambda \in L} B_\lambda^\uparrow\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_\lambda}^S\right)^{-1}(B_\lambda)\right)\right)^\uparrow$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} B_\lambda^\uparrow := \bigcap_{\lambda \in L} B_\lambda^\uparrow$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E, S) and (F, S'), by $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^S$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^S$ we have $\emptyset^S \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{B_\lambda^\uparrow\right\}_{\lambda\in L}$ is defined by the de Morgan law $\bigvee_{\lambda\in L} B_\lambda^\uparrow = \neg\left(\bigwedge_{\lambda\in L}\neg\left(B_\lambda^\uparrow\right)\right)$. Typically $\bigvee_{\lambda\in L} B_\lambda^\uparrow \subsetneqq \bigcup_{\lambda\in L} B_\lambda^\uparrow$, and if all B_λ are nonempty we have that $\bigvee_{\lambda\in L} B_\lambda^\uparrow = \bigcup_{\lambda\in L} B_\lambda^\uparrow$ holds if and only if all the $B_\lambda^\uparrow$ have the same base-space.

For each individual $i \in I$ there is a *possibility correspondence* $\Pi_i : \Sigma \to 2^\Sigma$ with the following properties:

0. *Confinedness:* If $s \in S$ then $\Pi_i(s) \subseteq S'$ for some $S' \preceq S$.
1. *Generalized Reflexivity:* $s \in \Pi_i^\uparrow(s)$ for every $s \in \Sigma$.[2]
2. *Stationarity:* $s' \in \Pi_i(s)$ implies $\Pi_i(s') = \Pi_i(s)$.
3. *Projections Preserve Awareness:* If $s \in S'$, $s \in \Pi_i(s)$ and $S \preceq S'$ then $s_S \in \Pi_i(s_S)$.
4. *Projections Preserve Ignorance:* If $s \in S'$ and $S \preceq S'$ then $\Pi_i^\uparrow(s) \subseteq \Pi_i^\uparrow(s_S)$.
5. *Projections Preserve Knowledge:* If $S \preceq S' \preceq S''$, $s \in S''$ and $\Pi_i(s) \subseteq S'$ then $(\Pi_i(s))_S = \Pi_i(s_S)$.

Generalized Reflexivity and Stationarity are the analogues of the partitional properties of the possibility correspondence in partitional information structures. In particular, Generalized Reflexivity will yield the truth property (that what an individual knows indeed obtains); Stationarity will guarantee the introspection properties (that an individual knows what she knows and that an individual knows what she ignores provided she is aware of it).

Properties 3. to 5. guarantee the coherence of the knowledge and the awareness of individuals down the lattice structure. They compare the possibility sets of an individual in a state s and its projection s_S. The properties guarantee that, first, at the projected state s_S the individual knows nothing she does not know at s, and second, at the projected state s_S the individual is not aware of anything she is unaware of at s (Projections Preserve Ignorance). Third, at the projected state s_S the individual knows every event she knows at s, provided that this event is based in a space lower than or equal to S (Projections Preserve Knowledge). Fourth, at the projected state s_S the individual is aware of every event she is aware of at s, provided that this event is based in a space lower than or equal to S (Projections Preserve Awareness).

[2]Here and in what follows, we abuse notation slightly and write $\Pi_i^\uparrow(\omega)$ for $(\Pi_i(\omega))^\uparrow$.

Definition 1. *The tuple*

$$\underline{\Sigma} := \left\langle (S_\alpha)_{\alpha\in\mathcal{A}}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta\preceq S_\alpha}, (\Pi_i)_{i\in I}\right\rangle,$$

is called an unawareness structure *for the set of individuals I.*

Definition 2. *The* knowledge operator *of individual i on events E is defined, as usual, by $K_i(E) := \{s \in \Sigma : \Pi_i(s) \subseteq E\}$, if there is a state s such that $\Pi_i(s) \subseteq E$, and by $K_i(E) := \emptyset^{S(E)}$ otherwise.*

The following proposition is proved in Heifetz, Meier and Schipper (2006):

Proposition 1. *If E is an event, then $K_i(E)$ is an $S(E)$-based event.*

Definition 3. *The* unawareness operator *of individual i from events to events is defined by $U_i(E) = \neg K_i(E) \cap \neg K_i \neg K_i(E)$, and the* awareness operator *is then naturally defined by $A_i(E) = \neg U_i(E)$.*

This is the Modica-Rustichini (1999) definition.

By Proposition 1 and the definition of the negation, we have $A_i(E) = K_i(E) \cup K_i\neg K_i(E)$.

For further properties of the unawareness structure see the complete paper.

4 The Canonical Unawareness Structure

Where do unawareness structures come from? What formal properties of knowledge and awareness do they capture? And can one model by unawareness structures *all* situations with such properties?

In this section we shall approach these questions by using the logical apparatus from section 2. "Properties" will be expressed by formulas of the syntax defined there; a "situation" will be a description of properties (a set of formulas) which is both *consistent* in the system (i.e. does not entail a contradiction), and *comprehensive* (i.e., for each potential property of knowledge and awareness, the description contains either the property or its negation).

We shall show that the collection of all such descriptions of situations does indeed constitute an unawareness structure (theorem 1 below). This unawareness structure is called the *canonical unawareness structure.* Any formula that belongs to some state of the canonical structure, and expresses knowledge or awareness, is mirrored accurately by a corresponding property of knowledge or awareness of *events* in the structure, a property that obtains in that state (theorem 2 below).

In short, the canonical unawareness structure consists of all the consistent and comprehensive descriptions of mutual knowledge and awareness, and these explicit descriptions are reflected by the knowledge and awareness operators on events in this structure. This substantiates the use of unawareness structure by providing an adequate foundation, and addresses the questions posed at the beginning of this section.

We now proceed with the definition of the canonical unawareness structure. Recall that X is the set of primitive propositions. For a subset $\alpha \subseteq X$, let Ω_α be the set of maximally consistent sets ω_α of formulas in the sub-language $\mathcal{L}_\alpha$, that is, ω_α is consistent and for every formula $\varphi \in \mathcal{L}_\alpha \backslash \omega_\alpha$, the set $\omega_\alpha \cup \{\varphi\}$ is inconsistent. Let $\Omega = \bigcup_{\alpha \subseteq X} \Omega_\alpha$. We define $\Omega_\beta \preceq \Omega_\alpha$ whenever $\beta \subseteq \alpha$.

This makes $\{\Omega_\alpha\}_{\alpha \subseteq X}$ into a lattice of spaces. To complete its definition as *the canonical unawareness structure*, we proceed by defining the projections among the spaces and the possibility correspondences of individuals.

Lemma 1. *Let $\alpha \subseteq X$ and $\Gamma \subseteq \mathcal{L}_\alpha$. If Γ is a consistent subset of $\mathcal{L}_\alpha$ then it can be extended to a maximally consistent subset ω_α of $\mathcal{L}_\alpha$.*

If $\Omega_\alpha \succeq \Omega_\beta$ (i.e., $\alpha \supseteq \beta$) we define a projection $r^\alpha_\beta : \Omega_\alpha \longrightarrow \Omega_\beta$ by $r^\alpha_\beta(\omega) := \omega \cap \mathcal{L}_\beta$.

Proposition 2. *The projection r^α_β is well-defined and surjective.*

Remark 1. *If $\alpha \supseteq \beta \supseteq \gamma$ then $r^\alpha_\gamma = r^\beta_\gamma \circ r^\alpha_\beta$.*

For every formula φ, denote $[\varphi] := \{\omega \in \Omega : \varphi \in \omega\}$.

Proposition 3. *$[\varphi]$ is an event.*

Definition 4. *For every $\omega \in \Omega$ and $i \in I$ define the possibility set*

$$\Pi_i(\omega) = \left\{ \omega' \in \Omega : \begin{array}{l} \text{For every formula } \varphi \\ \text{(i) } k_i\varphi \in \omega \text{ implies } \varphi \in \omega' \\ \text{(ii) } a_i\varphi \in \omega \text{ iff } (\varphi \in \omega' \text{ or } \neg\varphi \in \omega') \end{array} \right\}.$$

Definition 5. *For every $\omega \in \Omega$ and $i \in I$ define* $\alpha(\omega, i) := \{x \in X \mid a_i(x) \in \omega\}$.

Proposition 4. *For every $\omega \in \Omega$ and $i \in I$ we have* $\omega \cap \mathcal{L}_{\alpha(\omega,i)} \in \Pi_i(\omega)$.

Theorem 1. *For every $\omega \in \Omega$ and $i \in I$, $\Pi_i(\omega)$ is nonempty and satisfies the properties 0. - 5. of a possibility correspondence.*

By Remark 1, Proposition 2 and Theorem 1 it follows that

Corollary 1. *The tuple*

$$\underline{\Omega} := \left\langle (\Omega_\alpha)_{\alpha \subseteq X}, \left(r^\alpha_\beta\right)_{\beta \subseteq \alpha \subseteq X}, (\Pi_i)_{i \in I} \right\rangle,$$

is an unawareness structure for the set of individuals I.

The definition of the canonical unawareness structure is hence complete. We now proceed to show that the internal structure of its states is indeed reflected by operations on events in the structure. In particular, knowledge as expressed in syntactic terms within a state ($k_i\varphi$) gets translated to knowledge of the corresponding event ($K_i[\varphi]$).

Theorem 2. *For $\varphi \in \mathcal{L}$, we have*

$$\begin{aligned} [\neg\varphi] &= \neg[\varphi], \\ [\varphi \wedge \psi] &= [\varphi] \cap [\psi], \\ [k_i\varphi] &= K_i[\varphi]. \end{aligned}$$

It follows that for $\varphi \in \mathcal{L}$, we have $A_i[\varphi] = [a_i\varphi]$ and $U_i[\varphi] = [u_i\varphi]$.

4.1 Semantics, Soundness and Completeness

In this section we have to show how the canonical unawareness structure manifests that our axiom system is strongly sound and strongly complete with respect the family of unawareness structures. Strong soundness means that if a formula φ is provable from a set of formulas Γ (with the same set of atomic propositions as in φ), then φ obtains in all states of all unawareness structures in which Γ obtains. Strong completeness means the reverse implication: If φ obtains in all states of all unawareness structures in which such a Γ obtains, then φ is provable from Γ. Strong soundness and strong completeness is thus another way to formulate the sense in which the axiom system provides a foundation and substantiates the notion of an unawareness structure.

Strong completeness means that syntactic implication follows from semantic implication. Strong completeness is equivalent to that every consistent set of formulas is true in some state in some unawareness structure.

After having defined evaluation functions $v : X \longrightarrow \mathcal{E}$, where $\mathcal{E}$ is the set of events; unawareness models $\underline{\Sigma}^v := (\underline{\Sigma}, v)$; the model relation $(\underline{\Sigma}^v, s) \models \varphi$, where s is a state in Σ; and the notion of semantic implication $\Gamma \models \varphi$ (for formal definitions, see the complete article); we can formulate and prove the following strong characterization Theorem. It says that the notions of semantic and syntactic implication coincide:

Theorem 3. (Strong soundness and strong completeness) *For $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, we have $\Gamma \vdash \varphi$*

if and only if $\Gamma \models \varphi$*. Furthermore,* $\Gamma \vdash \varphi$ *if and only if in the canonical unawareness model* $\underline{\Omega}$*, for every* $\omega \in \Omega_\alpha$ *such that* $\Pr(\varphi) \subseteq \alpha$*, we have that* $(\underline{\Omega}, \omega) \models \Gamma$ *implies* $(\underline{\Omega}, \omega) \models \varphi$*.*

Corollary 2. *A set of formulas is consistent if and only if it has a model.*

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