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# Dealing With Logical Omniscience

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**Joseph Y. Halpern**  
Cornell University  
Ithaca, NY 14853 USA  
halpern@cs.cornell.edu

**Riccardo Pucella**  
Northeastern University  
Boston, MA 02115 USA  
riccardo@ccs.neu.edu

## Abstract

We examine four approaches for dealing with the logical omniscience problem and their potential applicability: the syntactic approach, awareness, algorithmic knowledge, and impossible possible worlds. Although in some settings these approaches are equi-expressive and can capture all epistemic states, in other settings of interest they are not. In particular, adding probabilities to the language allows for finer distinctions between different approaches.

## 1 Introduction

Logics of knowledge based on possible-world semantics are useful in many areas of knowledge representation and reasoning, ranging from security to distributed computing to game theory. In these models, an agent is said to know a fact  $\varphi$  if  $\varphi$  is true in all the worlds she considers possible. While reasoning about knowledge with this semantics has proved useful, as is well known, it suffers from what is known as the *logical omniscience* problem: under possible-world semantics, agents know all tautologies and know the logical consequences of their knowledge.

While logical omniscience is certainly not always an issue, in many applications it is. For example, in the context of distributed computing, we are interested in polynomial-time algorithms, although in some cases the knowledge needed to perform optimally may require calculations that cannot be performed in polynomial time (unless  $P=NP$ ) [Moses and Tuttle 1988]; in the context of security, we may want to reason about computationally bounded adversaries who cannot factor a large composite number, and thus cannot be logically omniscient; in game theory, we may be interested in the impact of computational resources on solution concepts (e.g., what will agents do if computing a Nash equilibrium is difficult).

Not surprisingly, many approaches for dealing with the log-

ical omniscience problem have been suggested (see [Fagin, Halpern, Moses, and Vardi 1995, Chapter 9] and [Moreno 1998]). A far from exhaustive list of approaches includes:

- *syntactic approaches* [Eberle 1974; Moore and Hendrix 1979; Konolige 1986], where an agent's knowledge is represented by a set of formulas (intuitively, the set of formulas she knows);
- *awareness* [Fagin and Halpern 1988], where an agent knows  $\varphi$  if she is aware of  $\varphi$  and  $\varphi$  is true in all the worlds she considers possible;
- *algorithmic knowledge* [Halpern, Moses, and Vardi 1994] where, roughly speaking, an agent knows  $\varphi$  if her knowledge algorithm returns "Yes" on a query of  $\varphi$ ; and
- *impossible worlds* [Rantala 1982], where the agent may consider possible worlds that are logically inconsistent (for example, where  $p$  and  $\neg p$  are both true).

Which approach is best to use, of course, depends on the application. Our goal is to elucidate the aspects of the application that make a logic more or less appropriate. We focus here on the expressive power of these approaches. It may seem that there is not much to say with regard to expressiveness, since it has been shown that all these approaches are equi-expressive and, indeed, can capture all epistemic states (see [Wansing 1990; Fagin, Halpern, Moses, and Vardi 1995] and Section 2). However, this result holds only if we allow an agent to consider no worlds possible. As we show, this equivalence no longer holds in contexts where agents must consider some worlds possible. This is particularly relevant with probability in the picture.

Expressive power is only part of the story. In the full version of this paper [Halpern and Pucella 2007], we consider (mainly by example) the *pragmatics* of dealing with logical omniscience—an issue that has largely been ignored: how to choose an approach and construct an appropriate model. Also for reasons of space, proofs of our technical results have been omitted, and can be found in the full paper.

## 2 The Four Approaches: A Review

We now review the standard possible-worlds approach and the four approaches to dealing logical omniscience discussed in the introduction. For ease of exposition we focus on the single-agent propositional case. While in many applications it is important to consider more than one agent and to allow first-order features, the issues that arise in dealing with these extensions are largely orthogonal to those involved in dealing with logical omniscience. Thus, we do not discuss these extensions here.

### 2.1 The Standard Approach

Starting with a set  $\Phi$  of propositional formulas, we close off under conjunction, negation, and the  $K$  operator. Call the resulting language  $\mathcal{L}^K$ . We give semantics to these formulas using Kripke structures. For simplicity, we focus on approaches that satisfy the K45 axioms (as well as KD45 and S5). In this case, a *K45 Kripke structure* is a triple  $(W, W', \pi)$ , where  $W$  is a nonempty set of *possible worlds* (or *worlds*, for short),  $W' \subseteq W$  is the set of worlds that the agent considers possible, and  $\pi$  is an *interpretation* that associates with each world a truth assignment  $\pi(w)$  to the primitive propositions in  $\Phi$ . Note that the agent need not consider every possible world (that is, each world in  $W$ ) possible. Then we have

$$(M, w) \models p \text{ iff } \pi(w)(p) = \mathbf{true} \text{ if } p \in \Phi.$$

$$(M, w) \models \neg\varphi \text{ iff } (M, w) \not\models \varphi.$$

$$(M, w) \models \varphi \wedge \psi \text{ iff } (M, w) \models \varphi \text{ and } (M, w) \models \psi.$$

$$(M, w) \models K\varphi \text{ iff } (M, w') \models \varphi \text{ for all } w' \in W'.$$

This semantics suffers from the logical omniscience problem. In particular, one sound axiom is  $(K\varphi \wedge K(\varphi \Rightarrow \psi)) \Rightarrow K\psi$ , which says that an agent's knowledge is closed under implication. In addition, the *knowledge generalization* inference rule is sound: From  $\varphi$  infer  $K\varphi$ . Thus, agents know all tautologies. As is well known, two other axioms are sound in K45 Kripke structures:  $K\varphi \Rightarrow KK\varphi$  and  $\neg K\varphi \Rightarrow K\neg K\varphi$ . These are known respectively as the positive and negative introspection axioms. (These properties characterize K45.)

In the structures we consider, we allow  $W'$  to be empty, in which case the agent does not consider any worlds possible. In such structures, the formula  $K(\text{false})$  is true. A *KD45 Kripke structure* is a K45 Kripke structure  $(W, W', \pi)$  where  $W' \neq \emptyset$ . Thus, in a KD45 Kripke structure, the agent always considers at least one world possible. In KD45 Kripke structures, the axiom  $\neg K(\text{false})$  is sound, which implies that the agent cannot know inconsistent facts. The logic KD45 results when we add this axiom to K45. *S5 Kripke structures* are KD45 Kripke structures

where  $W = W'$ ; that is, the agent considers all worlds in  $W$  possible. In S5 Kripke structures, the axiom  $K\varphi \Rightarrow \varphi$ , which says that the agent can know only true facts, is sound. Adding this axiom to the KD45 axioms gives us the logic S5.

### 2.2 The Syntactic Approach

The intuition behind the syntactic approach for dealing with logical omniscience is simply to explicitly list, at every possible world  $w$ , the set of formulas that the agent knows at  $w$ . A *syntactic structure* has the form  $M = (W, W', \pi, \mathcal{C})$ , where  $(W, W', \pi)$  is a K45 Kripke structure and  $\mathcal{C}$  associates a set of formulas  $\mathcal{C}(w)$  with every world  $w \in W$ . The semantics of primitive propositions, conjunction, and negation is just the same as for Kripke structures. For knowledge, we have

$$(M, w) \models K\varphi \text{ iff } \varphi \in \mathcal{C}(w).$$

### 2.3 Awareness

Awareness is based on the intuition that an agent should be aware of a concept before she can know it. The formulas that an agent is aware of are represented syntactically; we associate with every world  $w$  the set  $\mathcal{A}(w)$  of formulas that the agent is aware of. For an agent to know a formula  $\varphi$ , not only does  $\varphi$  have to be true at all the worlds she considers possible, but she has to be aware of  $\varphi$  as well. A *K45 awareness structure* is a tuple  $M = (W, W', \pi, \mathcal{A})$ , where  $(W, W', \pi)$  is a K45 Kripke structure and  $\mathcal{A}$  maps worlds to sets of formulas. We now define

$$(M, w) \models K\varphi \text{ iff } (M, w') \models \varphi \text{ for all } w' \in W' \text{ and } \varphi \in \mathcal{A}(w).^1$$

We can define KD45 and S5 awareness structures in the obvious way:  $M = (W, W', \pi, \mathcal{A})$  is a KD45 awareness structure when  $(W, W', \pi)$  is a KD45 structure, and an S5 awareness structure when  $(W, W', \pi)$  is an S5 structure.

### 2.4 Algorithmic Knowledge

In some applications, there is a computational intuition underlying what an agent knows; that is, an agent computes what she knows using an algorithm. *Algorithmic knowledge* is one way of formalizing this intuition. An *algorithmic knowledge structure* is a tuple  $M = (W, W', \pi, \mathcal{A})$ , where  $(W, W', \pi)$  is a K45 Kripke structure and  $\mathcal{A}$  is a *knowledge algorithm* that returns “Yes”, “No”, or “?” given

<sup>1</sup>In [Fagin and Halpern 1988], the symbol  $K$  is reserved for the standard definition of knowledge; the definition we have just given is denoted as  $X\varphi$ , where  $X$  stands for *explicit* knowledge. A similar remark applies to the algorithmic knowledge approach below. We use  $K$  throughout for ease of exposition.

a formula  $\varphi$ .<sup>2</sup> Intuitively,  $A(\varphi)$  returns “Yes” if the agent can compute that  $\varphi$  is true, “No” if the agent can compute that  $\varphi$  is false, and “?” otherwise. In algorithmic knowledge structures,

$$(M, w) \models K\varphi \text{ iff } A(\varphi) = \text{“Yes”}.$$

An important class of knowledge algorithms consists of the *sound* knowledge algorithms. When a sound knowledge algorithm returns “Yes” to a query  $\varphi$ , then the agent knows (in the standard sense)  $\varphi$ , and when it returns “No” to a query  $\varphi$ , then the agent does not know (again, in the standard sense)  $\varphi$ . Thus, if  $A$  is a sound knowledge algorithm, then  $A(\varphi) = \text{“Yes”}$  implies  $(M, w) \models \varphi$  for all  $w \in W'$ , and  $A(\varphi) = \text{“No”}$  implies there exists  $w \in W'$  such that  $(M, w) \models \neg\varphi$ . (When  $A(\varphi) = \text{“?”}$ , nothing is prescribed.)

Algorithmic knowledge can be seen as a generalization of a number of approaches in the literature, although they are not generally cast as algorithmic knowledge. Ramanujam [1999] defines an agent to know  $\varphi$  in a model if she can determine that  $\varphi$  is true in the submodel generated by the visible states (the part of the model that the agent sees, such as immediate neighbors in a distributed system), using the model-checking procedure for a standard logic of knowledge. In this case, the knowledge algorithm is simply the model-checking procedure. Another example is recent work on justification logics [Fitting 2005; Artemov and Nogina 2005], based on the intuition that an agent knows  $\varphi$  if she can prove that  $\varphi$  holds in some underlying constructive logic of proofs. The knowledge algorithm in this case consists of searching for a proof of  $\varphi$ .

## 2.5 Impossible Worlds

The impossible-worlds approach relies on relaxing the notion of possible world. Take the special case of logical omniscience that says that an agent knows all tautologies. This is a consequence of the fact that a tautology must be true at every possible world. Thus, one way to eliminate this problem is to allow tautologies to be false at some worlds. Clearly, those worlds do not obey the usual laws of logic—they are *impossible possible worlds* (or *impossible worlds*, for short).

A *K45* (resp., *KD45*, *S5*) *impossible-worlds structure* is a tuple  $M = (W, W', \pi, \mathcal{C})$ , where  $(W, W' \cap W, \pi)$  is a *K45* (resp., *KD45*, *S5*) Kripke structure,  $W'$  is the set of worlds that the agent considers possible, and  $\mathcal{C}$  associates with each world in  $W' - W$  a set of formulas.  $W'$ , the set of worlds the agent considers possi-

<sup>2</sup>In [Halpern, Moses, and Vardi 1994], the knowledge algorithm is also given an argument that describes the agent’s local state, which, roughly speaking, captures the relevant information that the agent has. However, in our single-agent static setting, there is only one local state, so this argument is unneeded.

ble, is not required to be a subset of  $W$ —the agent may well include impossible worlds in  $W'$ . The worlds in  $W' - W$  are the impossible worlds. We can also consider a class of impossible-worlds structures intermediate between *K45* and *KD45* impossible-worlds structures. A *KD45<sup>-</sup> impossible-worlds structure* is a *K45* impossible-worlds structure  $(W, W', \pi, \mathcal{C})$  where  $W'$  is nonempty. In a *KD45<sup>-</sup> impossible-worlds structure*, we do not require that  $W' \cap W$  be nonempty.

A formula  $\varphi$  is true at a world  $w \in W' - W$  if and only if  $\varphi \in \mathcal{C}(w)$ ; for worlds  $w \in W$ , the truth assignment is like that in Kripke structures. Thus,

- if  $w \in W$ , then  $(M, w) \models p$  iff  $\pi(w)(p) = \mathbf{true}$ ;
- if  $w \in W$ , then  $(M, w) \models K_i\varphi$  iff  $(M, w') \models \varphi$  for all  $w' \in W'$ ;
- if  $w \in W' - W$ , then  $(M, w) \models \varphi$  iff  $\varphi \in \mathcal{C}(w)$ .

We remark that when we speak of validity in impossible-worlds structures, we mean truth at all possible worlds in  $W$  in all impossible-worlds structures  $M = (W, \dots)$ .

## 3 Expressive Power

There is a sense in which all four approaches are equi-expressive, and can capture all states of knowledge.

**Theorem 3.1:** [Wansing 1990; Fagin, Halpern, Moses, and Vardi 1995] *For every finite set  $F$  of formulas and every propositionally consistent set  $G$  of formulas, there exists a syntactic structure (resp., *K45* awareness structure, *KD45<sup>-</sup> impossible-worlds structure*, *algorithmic knowledge structure*)  $M = (W, \dots)$  and a world  $w \in W$  such that  $(M, w) \models K\varphi$  if and only if  $\varphi \in F$ , and  $(M, w) \models \psi$  for all  $\psi \in G$ .*

Despite the name, the introspective axioms of *K45* are not valid in *K45* awareness structures or *K45* impossible-worlds structures. Indeed, it follows from Theorem 3.1 that no axioms of knowledge are valid in these structures. (Take  $F$  to be the empty set.)

As we now show, these structures support only propositional reasoning, which we can characterize by the following axiom:

$$\text{All substitution instances of valid formulas of propositional logic.} \quad (\text{Prop})$$

and the following inference rule:

$$\text{From } \varphi \Rightarrow \psi \text{ and } \varphi \text{ infer } \psi. \quad (\text{MP})$$

**Theorem 3.2:**  $\{Prop, MP\}$  is a sound and complete axiomatization of  $\mathcal{L}^K$  with respect to K45 awareness structures (resp., K45 and KD45<sup>-</sup> impossible-worlds structures, syntactic structures, algorithmic knowledge structures).

It follows from Theorem 3.2 that a formula is valid with respect to K45 awareness structures (resp., K45 and KD45<sup>-</sup> impossible-worlds structures, syntactic structures, algorithmic knowledge structures) if and only if it is propositionally valid, if we treat formulas of the form  $K\varphi$  as primitive propositions. Thus, deciding if a formula is valid is co-NP complete, just as it is for propositional logic.

Theorems 3.1 and 3.2 rely on the fact that we are considering K45 awareness structures and KD45<sup>-</sup> (or K45) impossible-worlds structures. (Whether we consider K45, KD45, or S5 is irrelevant in the case of syntactic structures and algorithmic knowledge structures, since the truth of a formula does not depend on what worlds an agent considers possible.) As we now show, there are constraints on what can be known if we consider KD45 and S5 awareness structures and impossible-worlds structures.

A set of formulas  $F$  is *downward closed* if the following conditions hold:

- (a) if  $\varphi \wedge \psi \in F$ , then both  $\varphi$  and  $\psi$  are in  $F$ ;
- (b) if  $\neg\neg\varphi \in F$ , then  $\varphi \in F$ ;
- (c) if  $\neg(\varphi \wedge \psi) \in F$ , then either  $\neg\varphi \in F$  or  $\neg\psi \in F$  (or both); and
- (d) if  $K\varphi \in F$ , then  $\varphi \in F$ .

We say that  $F$  is *k-compatible* with  $F'$  if  $K\psi \in F'$  implies that  $\psi \in F$ .

**Proposition 3.3:** Suppose that  $M = (W, W', \dots)$  is a KD45 awareness structure (resp., KD45 impossible-worlds structure),  $w \in W$ , and  $w' \in W'$  (resp.,  $w' \in W \cap W'$ ). Let  $F = \{\varphi \mid (M, w) \models K\varphi\}$  and let  $F' = \{\psi \mid (M, w') \models \psi\}$ . Then

- (a)  $F'$  is propositionally consistent downward-closed set of formulas that contains  $F$ ;
- (b) if  $M$  is a KD45 impossible-worlds structure then  $F$  is *k-compatible* with  $F'$ .

The next result show that the constraints on  $F$  described in Proposition 3.3 are the only constraints on  $F$ .

**Theorem 3.4:** If  $F$  and  $F'$  are such that  $F'$  is propositionally consistent downward-closed set of formulas that contains  $F$ , then there exists a KD45 awareness structure  $M = (\{w, w'\}, \{w'\}, \pi, \mathcal{A})$  such that  $(M, w) \models K\varphi$  iff  $\varphi \in F$  and  $(M, w') \models \psi$  for all  $\psi \in F'$ . If, in addition,  $F$  is *k-compatible* with  $F'$ , then

there exists a KD45 impossible-worlds structure  $M = (\{w, w'\}, \{w', w''\}, \pi, \mathcal{C})$  such that  $(M, w) \models K\varphi$  iff  $\varphi \in F$  and  $(M, w') \models \psi$  for all  $\psi \in F'$ . Finally, if  $F = F'$ , then we can take  $w = w'$ , so that  $M$  is an S5 awareness (resp., S5 impossible-worlds) structure.

We can characterize these properties axiomatically. Let  $(Ver)$  (for *Veridicality*) be the standard axiom that says that everything known must be true:

$$K\varphi \Rightarrow \varphi. \quad (Ver)$$

Let  $AX_{Ver}$  be the axiom system consisting of  $\{Prop, MP, Ver\}$ . The fact that the set of formulas known must be a subset of a downward closed set is characterized by the following axiom:

$$\begin{aligned} &\neg(K\varphi_1 \wedge \dots \wedge K\varphi_m) \\ &\text{if } AX_{Ver} \vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_n). \end{aligned} \quad (DC)$$

The key point here is that, as we shall show, a propositionally consistent set of formulas that is downward closed must be consistent with  $AX_{Ver}$ .

The fact that the set of formulas that is known is *k-compatible* with a downward closed set of formulas is characterized by the following axiom:

$$\begin{aligned} &(K\varphi_1 \wedge \dots \wedge K\varphi_n) \Rightarrow (K\psi_1 \vee \dots \vee K\psi_m) \\ &\text{if } AX_{Ver} \vdash \varphi_1 \wedge \dots \wedge \varphi_n \Rightarrow (K\psi_1 \vee \dots \vee K\psi_m). \end{aligned} \quad (KC)$$

Axiom  $DC$  is just the special case of axiom  $KC$  where  $m = 0$ . Note that  $KC$  (and therefore  $DC$ ) follow from  $Ver$ . Let  $AX_{DC} = \{Prop, MP, DC\}$  and let  $AX_{KC} = \{Prop, MP, KC\}$ .

**Theorem 3.5:**

- (a)  $AX_{DC}$  is a sound and complete axiomatization of  $\mathcal{L}^K$  with respect to KD45 awareness structures;
- (b)  $AX_{KC}$  is a sound and complete axiomatization of  $\mathcal{L}^K$  with respect to KD45 impossible-worlds structures;
- (c)  $AX_{Ver}$  is a sound and complete axiomatization of  $\mathcal{L}^K$  with respect to S5 awareness structures and S5 impossible-worlds structures.

**Corollary 3.6:** The satisfiability problem for the language  $\mathcal{L}^K$  with respect to KD45 awareness structures (resp., KD45 impossible-worlds structures, S5 awareness structures) is NP-complete.

## 4 Adding Probability

While the differences between K45, KD45<sup>-</sup>, and KD45 impossible-worlds structures may appear minor, they turn out to be important when we add probability to the picture.

As pointed out by Cozic [2005], standard models for reasoning about probability suffer from the same logical omniscience problem as models for knowledge. In the language considered by Fagin, Halpern, and Megiddo [1990] (FHM from now on), there are formulas that talk explicitly about probability. A formula such as  $\ell(\text{Prime}_n) = 1/3$  says that the probability that  $n$  is prime is  $1/3$ . In the FHM semantics, a probability is put on the set of worlds that the agent considers possible. The probability of a formula  $\varphi$  is then the probability of the set of worlds where  $\varphi$  is true. Clearly, if  $\varphi$  and  $\psi$  are logically equivalent, then  $\ell(\varphi) = \ell(\psi)$  will be true. However, the agent may not recognize that  $\varphi$  and  $\psi$  are equivalent, and so may not recognize that  $\ell(\varphi) = \ell(\psi)$ . Problems of logical omniscience with probability can to some extent be reduced to problems of logical omniscience with knowledge in a logic that combines knowledge and probability [Fagin and Halpern 1994]. For example, the fact that an agent may not recognize  $\ell(\varphi) = \ell(\psi)$  when  $\varphi$  and  $\psi$  are equivalent just amounts to saying that if  $\varphi \Leftrightarrow \psi$  is valid, then we do not necessarily want  $K(\ell(\varphi) = \ell(\psi))$  to hold. However, adding knowledge and awareness does not prevent  $\ell(\varphi) = \ell(\psi)$  from holding. This is not really a problem if we interpret  $\ell(\varphi)$  as the objective probability of  $\varphi$ ; if  $\varphi$  and  $\psi$  are equivalent, it is an objective fact about the world that their probabilities are equal, so  $\ell(\varphi) = \ell(\psi)$  should hold. On the other hand, if  $\ell(\varphi)$  represents the agent's subjective view of the probability of  $\varphi$ , then we do not want to require  $\ell(\varphi) = \ell(\psi)$  to hold. This cannot be captured in all approaches.

To make this precise, we first clarify the logic we have in mind. Let  $\mathcal{L}^{K,QU}$  be  $\mathcal{L}^K$  extended with linear inequality formulas involving probability (called likelihood formulas), in the style of FHM. A likelihood formula is of the form  $a_1\ell(\varphi_1) + \dots + a_n\ell(\varphi_n) \geq c$ , where  $a_1, \dots, a_n$  and  $c$  are integers. (For ease of exposition, we restrict  $\varphi_1, \dots, \varphi_n$  to be propositional formulas in likelihood formulas; however, the techniques presented here can be extended to deal with formulas that allow arbitrary nesting of  $\ell$  and  $K$ .) We give semantics to these formulas by extending Kripke structures with a probability distribution over the worlds that the agent considers possible. A *probabilistic KD45* (resp., *S5*) *Kripke structure* is a tuple  $(W, W', \pi, \mu)$ , where  $(W, W', \pi)$  is KD45 (resp., S5) Kripke structure, and  $\mu$  is a probability distribution over  $W'$ . To interpret likelihood formulas, we first define  $\llbracket \varphi \rrbracket_M = \{w \in W \mid \pi(w)(\varphi) = \text{true}\}$ , for a propositional formula  $\varphi$ . We then extend the semantics of  $\mathcal{L}^K$  with the following rule for interpreting likelihood formulas:

$$(M, w) \models a_1\ell(\varphi_1) + \dots + a_n\ell(\varphi_n) \geq c \text{ iff} \\ a_1\mu(\llbracket \varphi_1 \rrbracket_M \cap W') + \dots + a_n\mu(\llbracket \varphi_n \rrbracket_M \cap W') \geq c.$$

Note that the truth of a likelihood formula at a world does not depend on that world; if a likelihood formula is true at some world of structure  $M$ , it is true at every world of  $M$ .

FHM give an axiomatization for likelihood formulas in probabilistic structures. Aside from propositional reasoning axioms, one axiom captures reasoning with linear inequalities. A *basic inequality formula* is a formula of the form  $a_1x_1 + \dots + a_kx_k + a_{k+1} \leq b_1y_1 + \dots + b_my_m + b_{m+1}$ , where  $x_1, \dots, x_k, y_1, \dots, y_m$  are (not necessarily distinct) variables. A *linear inequality formula* is a Boolean combination of basic linear inequality formulas. A linear inequality formula is valid if the resulting inequality holds under every possible assignment of real numbers to variables. For example, the formula  $(2x + 3y \leq 5z) \wedge (x - y \leq 12z) \Rightarrow (3x + 2y \leq 17z)$  is a valid linear inequality formula. To get an instance of *Ineq*, we replace each variable  $x_i$  that occurs in a valid formula about linear inequalities by a likelihood term of the form  $\ell(\psi)$  (naturally, each occurrence of the variable  $x_i$  must be replaced by the same primitive expectation term  $\ell(\psi)$ ). (We can replace *Ineq* by a sound and complete axiomatization for Boolean combinations of linear inequalities; one such axiomatization is given in FHM.)

The other axioms of FHM are specific to probabilistic reasoning, and capture the defining properties of probability distributions:

$$\ell(\text{true}) = 1 \\ \ell(\neg\varphi) = 1 - \ell(\varphi) \\ \ell(\varphi \wedge \psi) + \ell(\varphi \wedge \neg\psi) = \ell(\varphi).$$

It is straightforward to extend all the approaches in Section 2 to the probabilistic setting. In this section, we only consider probabilistic awareness structures and probabilistic impossible-worlds structures, because the interpretation of both algorithmic knowledge and knowledge in syntactic structures does not depend on the set of worlds or any probability distribution over the set of worlds.

A *KD45* (resp., *S5*) *probabilistic awareness structure* is a tuple  $(W, W', \pi, \mathcal{A}, \mu)$  where  $(W, W', \pi, \mathcal{A})$  is a KD45 (resp., S5) awareness structure and  $\mu$  is a probability distribution over the worlds in  $W'$ . Similarly, a *KD45<sup>-</sup>* (resp., *S5*) *probabilistic impossible-worlds structure* is a tuple  $(W, W', \pi, \mathcal{C}, \mu)$  where  $(W, W', \pi, \mathcal{C})$  is a KD45<sup>-</sup> (resp., KD45, S5) impossible-worlds structure and  $\mu$  is a probability distribution over the worlds in  $W'$ . Since the set of worlds that are assigned probability must be nonempty, when dealing with probability, we must restrict to KD45 awareness structures and KD45<sup>-</sup> impossible-worlds structures, extended with a probability distribution over the set of worlds the agent considers possible. As we now show, adding probability to the language allows finer distinctions between awareness structures and impossible-worlds structures.

In probabilistic awareness structures, the axioms of probability described by FHM are all valid. For example,  $\ell(\varphi) = \ell(\psi)$  is valid in probabilistic awareness structures if  $\varphi$  and  $\psi$  are equivalent formulas. Using argu-

ments similar to those in Theorem 3.4, we can show that  $\neg K(\neg(\ell(\varphi) = \ell(\psi)))$  is valid in probabilistic awareness structures. Similarly, since  $\ell(\varphi) + \ell(\neg\varphi) = 1$  is valid in probability structures,  $\neg K(\neg(\ell(\varphi) + \ell(\neg\varphi) = 1))$  is valid in probabilistic awareness structures.

We can characterize properties of knowledge and likelihood in probabilistic awareness structures axiomatically. Let  $Prob$  denote a substitution instance of a valid formula in probabilistic logic (using the FHM axiomatization). By the observation above,  $Prob$  is sound in probabilistic awareness structures. Our reasoning has to take this into account. There is also an axiom  $KL$  that connects knowledge and likelihood:

$$K\varphi \Rightarrow \ell(\varphi) > 0. \quad (KL)$$

Let  $AX_{Ver}^P$  denote the axiom system consisting of  $\{Prop, MP, Prob, KL, Ver\}$ .

Let  $DC^P$  be the following strengthening of  $DC$ , somewhat in the spirit of  $KC$ :

$$\begin{aligned} &(K\varphi_1 \wedge \dots \wedge K\varphi_n) \Rightarrow (\psi_1 \vee \dots \vee \psi_m) \\ \text{if } &AX_{Ver}^P \vdash \varphi_1 \wedge \dots \wedge \varphi_n \Rightarrow (\psi_1 \vee \dots \vee \psi_m) \quad (DC^P) \\ &\text{and } \psi_1, \dots, \psi_m \text{ are likelihood formulas.} \end{aligned}$$

Finally, even though  $Ver$  is not sound in KD45 probabilistic awareness structures, a weaker version, restricted to likelihood formulas, is sound, since there is a single probability distribution in probabilistic awareness structures. Let  $WVer$  be the following axiom:

$$K\varphi \Rightarrow \varphi \text{ if } \varphi \text{ is a likelihood formula.} \quad (WVer)$$

Let  $AX_{DC}^P = \{Prop, MP, Prob, DC^P, WVer, KL\}$  be the axiom system obtained by replacing  $DC$  in  $AX_{DC}$  by  $DC^P$  and adding  $Prob$ ,  $WVer$ , and  $KL$ .

**Theorem 4.1:**

- (a)  $AX_{DC}^P$  is a sound and complete axiomatization of  $\mathcal{L}^{K,QU}$  with respect to KD45 probabilistic awareness structures.
- (b)  $AX_{Ver}^P$  is a sound and complete axiomatization of  $\mathcal{L}^{K,QU}$  with respect to S5 probabilistic awareness structures.

Things change significantly when we move to probabilistic impossible-worlds structures. In particular,  $Prob$  is no longer sound. For example, even if  $\varphi \Leftrightarrow \psi$  is valid,  $\ell(\varphi) = \ell(\psi)$  is not valid, because we can have an impossible possible world with positive probability where both  $\varphi$  and  $\neg\psi$  are true. Similarly,  $\ell(\varphi) + \ell(\neg\varphi) = 1$  is not valid. Indeed, both  $\ell(\varphi) + \ell(\neg\varphi) > 1$  and  $\ell(\varphi) + \ell(\neg\varphi) < 1$  are both satisfiable in impossible-worlds structures: the former requires that there be an impossible possible world

that gets positive probability where both  $\varphi$  and  $\neg\varphi$  are true, while the latter requires an impossible possible world with positive probability where neither is true. As a consequence, it is not hard to show that both  $K\neg(\ell(\varphi) = \ell(\psi))$  and  $K(\neg(\ell(\varphi) + \ell(\neg\varphi) = 1))$  are satisfiable in such impossible-worlds structures.<sup>3</sup> In fact, the only constraint on probability in probabilistic impossible-worlds structures is that it must be between 0 and 1. This constraint is expressed by the following axiom  $Bound$ :

$$\ell(\varphi) \geq 0 \wedge \ell(\varphi) \leq 1. \quad (Bound)$$

We can characterize properties of knowledge and likelihood in probabilistic impossible-worlds structures axiomatically. Let  $AX_{imp}^B = \{Prop, MP, Ineq, Bound, KL, WVer\}$ . We can think of  $AX_{imp}^B$  as being the core of probabilistic reasoning in impossible-worlds structures. Let  $AX_{Ver}^B$  denote the axiom system  $\{Prop, MP, Ineq, Bound, Ver, KL\}$ . Let  $KC^P$  denote the following extension of  $KC$ :

$$\begin{aligned} &(K\varphi_1 \wedge \dots \wedge K\varphi_n) \Rightarrow (\psi_1 \vee \dots \vee \psi_m) \\ \text{if } &AX_{Ver}^B \vdash \varphi_1 \wedge \dots \wedge \varphi_n \Rightarrow (\psi_1 \vee \dots \vee \psi_m) \quad (KC^P) \\ &\text{and } \psi_j \text{ is either a likelihood formula} \\ &\text{or of the form } K\psi', \text{ for } j = 1, \dots, m. \end{aligned}$$

Here again,  $DC^P$  is a special case of  $KC^P$ . Let  $AX_{KC}^B = \{Prop, MP, Bound, KC^P, WVer, KL\}$  obtained by replacing  $KC$  in  $AX_{KC}$  by  $KC^P$  and adding  $Bound$ ,  $WVer$ , and  $KL$ .

**Theorem 4.2:**

- (a)  $AX_{imp}^B$  is a sound and complete axiomatization of  $\mathcal{L}^{K,QU}$  with respect to KD45<sup>-</sup> probabilistic impossible-worlds structures.
- (b)  $AX_{KC}^B$  is a sound and complete axiomatization of  $\mathcal{L}^{K,QU}$  with respect to KD45 probabilistic impossible-worlds structures.
- (c)  $AX_{Ver}^B$  is a sound and complete axiomatization of  $\mathcal{L}^{K,QU}$  with respect to S5 probabilistic impossible-worlds structures with probabilities.

Observe that Theorem 4.2 is true even though probabilities are standard in impossible worlds: the probabilities of worlds still sum to 1. It is just the truth assignment to formulas that behaves in a nonstandard way in impossible worlds. Intuitively, while the awareness approach is modeling certain consequences of resource-boundedness in the

<sup>3</sup>We remark that Cozic [2005], who considers the logical omniscience problem in the context of probabilistic reasoning, makes somewhat similar points. Although he does not formalize things quite the way we do, he observes that, in his setting, impossible-worlds structures seem more expressive than awareness structures.

context of knowledge, it does not do so for probability. On the other hand, the impossible-worlds approach seems to extend more naturally to accommodate the consequences of resource-boundedness in probabilistic reasoning.

**Corollary 4.3:** *The satisfiability problem for the language  $\mathcal{L}^{K,QU}$  with respect to KD45 probabilistic awareness structures (resp., S5 probabilistic awareness structures, KD45<sup>-</sup> probabilistic impossible-worlds structures, KD45 probabilistic impossible worlds structures, S5 probabilistic impossible-worlds structures) is NP-complete.*

## 5 Conclusion

Many solutions have been proposed to the logical omniscience problem, differing as to the intuitions underlying the lack of logical omniscience. There has been comparatively little work on comparing approaches. We have attempted to do so here, focussing essentially on expressiveness for four popular approaches.

In comparing the expressive power of the approaches, we started with the well-known observation that the approaches are equi-expressive in the propositional case. However, this observation is true only if we allow the agent not to consider any world possible. If we require that at least one world be possible, then we get a difference in expressive power. This is particularly relevant when we have probabilities, because there has to be at least one world over which to assign probability. Indeed, when considering logical omniscience in the presence of probability, there can be quite significant differences in expressive power between the approaches, particularly awareness and impossible worlds.

As we said, in the full paper, we also consider the pragmatics of logical omniscience. Even in settings where the four approaches are equi-expressive, they model lack of logical omniscience quite differently. We thus have to deal with different issues when attempting to use one of them in practice. For example, if we are using a syntactic structure to represent a given situation, we need to explain where the function  $\mathcal{C}$  is coming from; with an awareness structure, we must explain where the awareness function is coming from; with an algorithmic knowledge structure, we must explain where the algorithm is coming from; and with an impossible-worlds structure, we must explain what the impossible worlds are. In some domains, there may be a natural interpretation for the awareness function, but it finding a natural impossible-worlds interpretation may be difficult; in other domains, the situation may be just the opposite. Given the increasing understanding of the importance of awareness in game-theoretic applications (see, for example, [Heifetz, Meier, and Schipper 2003; Halpern and Rêgo 2006a; Halpern and Rêgo 2006b]), these pragmatic issues assume more significance, and deserve further exploration.

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