The semantics of preference-based belief operators

(Extended abstract)*

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1 Introduction

In epistemic analyses of games, it is common to use subjective belief operators. There are numerous examples where KD45 operators like 'belief with probability 1' (e.g., Tan & Werlang [26]), 'belief with primary probability 1' (Brandenburger [10]) and 'conditional belief with probability 1' (Ben-Porath [7]) are applied. More recently, Brandenburger & Keisler [13] and Battigalli & Siniscalchi [6] have proposed non-monotonic subjective belief operators called 'assumption' and 'strong belief', respectively. These operators all have in common that they are based on subjective probabilities — arising from a probability distribution, a lexicographic probability system, or a conditional probability system — that represent the preferences of the player as a decision maker. Thus, in game theory there is a prevalence of preference-based belief operators.

While all the above contributions use subjective probabilities to define the epistemic operators, Morris [20] observes that it is unnecessary to go via subjective probabilities to derive subjective belief operators from the preferences of a decision maker. This suggestion has been followed by Asheim & Dufwenberg [4] and Asheim [3], who consider epistemic conditions for forward induction and backward inductions without the use of subjective probabilities. In the case of Asheim & Dufwenberg's [4] it is

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necessary for the characterization of forward induction to use incomplete preferences that cannot be represented by subjective probabilities, while Asheim [3] points to the possibility of characterizing backward induction without the use of subjective probabilities since one can convincingly argue that subjective probabilities play no role in the backward induction argument.

When deriving belief operators from preferences, it is essential that the preferences determine 'subjective possibility' (so that it can be determined whether an event is subjectively impossible) as well as 'epistemic priority' (so that one allows for non-trivial belief revision). As we shall see, preferences need not satisfy completeness to determine 'subjective possibility' and 'epistemic priority'. We intend to show how belief operators corresponding to those used in the literature can be derived from preferences that need not be complete.[1]

After presenting the decision-theoretic framework in Sect. 2, we show in Sect. 3 how a binary accessibility relation of epistemic priority Q can be derived from preferences that satisfy conditions that are weaker than those usually applied in the Anscombe–Aumann [2] framework. The properties of this priority relation are similar to but more general than those found, e.g., in Lamarre & Shoham [19] and Stalnaker [24, 25] in that reflexivity is not required. Furthermore, we show how preferences give rise to a vector of nested binary accessibility relations $(R_1, \ldots, R_L)$, where, for each k, R_k fulfills the usual properties of Kripke representations of beliefs; i.e., they are serial, transitive and euclidean. Finally, we establish that the two kinds of accessibility relations yield two equivalent representations of 'subjective possibility' and 'epistemic priority'.

In Sect. 4 we use Q to define the following belief operators:

- *Certain belief* corresponds to what Morris [20] calls 'Savage-belief' and means that the complement of the event is subjectively impossible.
- *Conditional belief* is a generalization of 'conditional belief with probability 1'.
- *Full belief* corresponds to what Stalnaker [25] calls 'absolutely robust belief'.

[1]'Epistemic priority' will here be used to refer to what elsewhere is sometimes referred to as 'plausibility' or 'prejudice'; see, e.g., Friedman & Halpern [15] and Lamerre & Shoham [19]. This is similar to 'preference' among states (or worlds) in nonmonotonic logic (cf. Shoham [23] and Kraus et al. [18]), leading agents towards some states and away from others. In contrast, we use 'preferences' in the decision-theoretic sense of a binary relation on the set of functions ('acts') from states to outcomes; see Sect. 2.

We then in Sect. 5 show how these operators can be characterized by means of $(R_1, \ldots, R_L)$, while we in Sect. 6 show that the full belief operator (while poorly behaved) is bounded by certain and conditional belief, which are KD45 operators.

In Sect. 7 we interpret our one-agent decision-theoretic framework in terms of the n-agent decision-theoretic framework encountered in games, and note how the characterization of full belief corresponds to the primitive definition of this operator in Asheim & Dufwenberg [4] as well as Brandenburger & Keisler's [13] concept of 'assumption'. In Sect. 8 we amend the decision-theoretic framework to be able to handle systems of conditional preferences used in analyses of extensive form games and show how Battigalli & Siniscalchi's [6] concept of 'strong belief' is related to full belief. We thereby reconcile and compare these non-standard notions of belief which have recently been used in epistemic analyses of games. We conclude in Sect. 9.

Throughout we assume that the subjective belief operators are derived from the initial (ex-ante) preferences of the decision maker. As this differs from Feinberg's [14] framework for subjective reasoning in dynamic games — where beliefs are not constrained to be evolving or revised, but are represented whenever there is a decision to be made — a closer comparison of our approach to his will be left for future work.

2 Decision-theoretic Framework

Consider a decision maker under uncertainty. Let W be a finite set of states (or possible worlds), where the decision maker is uncertain about what the true state is. Let Z be a finite set of outcomes. In the tradition of Anscombe & Aumann [2], the decision maker has preferences over the set of functions that assign an objective randomization over outcomes to any state. Any such function $\mathbf{x} : W \to \Delta(Z)$ is called an *act* on W. If the true state is a, then the preferences of the decision maker is a binary relation $\succeq^a$ on the set of acts, with $\succ^a$ and $\sim^a$ denoting the asymmetric and symmetric parts, respectively. For any $a \in W$, $\succeq^a$ is assumed to be *reflexive* and *transitive*, but not necessarily *complete*, and to satisfy *nontriviality* and *objective independence* (where these terms are defined in [8, p. 64]).

If $(\emptyset \neq)\ \phi \subseteq W$, let $\mathbf{x}_\phi$ denote the restriction of $\mathbf{x}$ to ϕ. Define the *conditional* binary relation $\succeq^a_\phi$ by $\mathbf{x}' \succeq^a_\phi \mathbf{x}''$ if, for some $\mathbf{y}$, $(\mathbf{x}'_\phi, \mathbf{y}_{\neg\phi}) \succeq^a (\mathbf{x}''_\phi, \mathbf{y}_{\neg\phi})$, where $\neg\phi$ denotes $W \backslash \phi$. By objective independence this definition does not depend on $\mathbf{y}$. Say that the state $b \in W$ is *Savage-null* at a if $\mathbf{x} \sim^a_{\{b\}} \mathbf{y}$ for all acts $\mathbf{x}$ and $\mathbf{y}$ on W. Say that b is deemed *infinitely more likely* than c at a ($b \gg^a c$; cf. [8, Def. 5.1]) if $b \neq c$, b

is not Savage-null at a, and $\mathbf{x} \succ^a_{\{b\}} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\{b,c\}} \mathbf{y}$. According to this definition, c may, but need not, be Savage-null at a if $b \gg^a c$. For any $a \in W$, $\succeq^a$ is assumed to satisfy *conditional completeness* — i.e., $\forall b \in W$, $\succeq^a_{\{b\}}$ is complete — *partitional priority* — i.e., if $b \gg^a c$, then, $\forall b' \in W$, $b \gg^a b'$ or $b' \gg^a c$ — *conditional continuity* (or "the conditional Archimedean property"), and *non-null state independence* (where the two latter terms are defined in [8, pp. 64–65]). Partitional priority can be shown to be redundant if conditional completeness is strengthened to completeness.

Let W be partitioned into equivalence classes, where $a \approx b$ denotes that a and b are in the same equivalence class, with $\approx$ being a reflexive, transitive and symmetric binary relation. Write $\tau^a := \{b \in W | \, a \approx b\}$. Let κ^a denote the set of states that are *not* Savage-null at a. Since $\succeq^a$ is nontrivial, $\kappa^a \neq \emptyset$. Assume that, for any $a \in W$, $\kappa^a \subseteq \tau^a$, and $\forall a' \in \tau^a$, $\mathbf{x} \succeq^{a'} \mathbf{y}$ iff $\mathbf{x} \succeq^a \mathbf{y}$. This assumption will ensure that the preference-based operators satisfy positive and negative introspection; it corresponds to "being aware of one's own type".

Definition 1 A *preference system* $\{\succeq^a | \, a \in W\}$ consists of

(1) a finite set of states W that is partitioned into equivalence classes by $\approx$, and

(2) for each $a \in W$, a reflexive and transitive binary relation $\succeq^a$ on the set of acts (where each act is a function $\mathbf{x} : W \to \Delta(Z)$ and Z is a finite set of outcomes), depending only on to which equivalence class a belongs, and satisfying nontriviality, objective independence, conditional completeness, conditional continuity, partitional priority, non-null state independence, and that $a \approx b$ if b is not Savage-null at a.

Say that $\succeq^a$ is *conditionally represented* by a vNM utility function $u^a : Z \to \mathbb{R}$ (writing $u^a(x) = \sum_{z \in Z} x(z) u^a(z)$ whenever $x \in \Delta(Z)$ is an objective randomization) if (1) $\succeq^a$ is nontrivial and (2) $\mathbf{x} \succeq^a_{\{b\}} \mathbf{y}$ iff $u^a(\mathbf{x}(b)) \geq u^a(\mathbf{y}(b))$ whenever b is not Savage-null at a. By the properties of Def. 1 it follows directly from the vNM theorem on expected utility representation that there, for any $a \in W$, exists a vNM utility function u^a such that $\succeq^a$ is conditionally represented by u^a. If $A \subseteq W$, say that $\mathbf{x}_A$ *weakly dominates* $\mathbf{y}_A$ at a if, $\forall b \in A$, $u^a(\mathbf{x}_A(b)) \geq u^a(\mathbf{y}_A(b))$, with strict inequality for some $c \in A$. Say that $\succeq^a$ *is admissible* on A if A is non-empty and $\mathbf{x} \succ^a \mathbf{y}$ whenever $\mathbf{x}_A$ weakly dominates $\mathbf{y}_A$ at a.

The following connection between admissibility on subsets and the infinitely-more-likely relation is important for relating the accessibility relations derived from preferences in the next section (cf. Prop. 4).

Proposition 1 *Let $A \neq \emptyset$ and $\neg A \neq \emptyset$. $\succeq^a$ is admissible on A iff $b \in A$ and $c \in \neg A$ imply $b \gg^a c$.*

3 From Preferences to Accessibility Relations

We derive two kinds of accessibility relation from preferences. The one kind is based on the infinitely-more-likely relation, while the other kind is based on admissibility on subsets.

Definition 2 aQb ("a does not have higher epistemic priority than b") if
(1) $a \approx b$,
(2) b is not Savage-null at a, and
(3) a is not deemed infinitely more likely than b at a.

Proposition 2 *The relation Q is serial,[2] transitive, and satisfies forward linearity[3] and quasi-backward linearity.[4]*

Consider the collection of all sets A satisfying that $\succeq^a$ is admissible on A. Since $\succeq^a$ is admissible on κ^a, it follows that the collection is non-empty as it is contains κ^a. Furthermore, since any $b \in A$ is not Savage-null at a if $\succeq^a$ is admissible on A, it follows that any set in this collection is a subset of κ^a. Finally, since $b \gg^a c$ implies that $c \gg^a b$ does not hold, it follows from Prop. 1 that $A' \subseteq A''$ or $A'' \subseteq A'$ if $\succeq^a$ is admissible on both A' and A'', implying that the sets in the collection are nested. Hence, there exists a vector of nested sets, $(\rho_1^a, \dots, \rho_{L^a}^a)$, on which $\succeq^a$ is admissible, satisfying: $\emptyset \neq \rho_1^a \subset \cdots \subset \rho_k^a \subset \cdots \subset \rho_{L^a}^a = \kappa^a \subseteq \tau^a$ (where $\subset$ denotes $\subseteq$ and $\neq$). Let $L := \max_{a \in W} L^a$. If, for some $a \in W$, $L^a < L$, let $\rho_{L^a}^a = \rho_k^a = \kappa^a$ for $k \in \{L^a + 1, \dots, L\}$. The collection of sets, $\{\rho_k^a | \, a \in W\}$, defines an accessibility relation, R_k.

Definition 3 aR_kb ("at a, b is deemed possible at the epistemic level k") if $b \in \rho_k^a$.

Proposition 3 *The vector of relations, $(R_1, \dots, R_L)$, has the following properties: For each $k \in \{1, \dots, L\}$, R_k is serial, transitive, and euclidian.[5] For each $k \in \{1, \dots, L-1\}$, (i) aR_kb implies $aR_{k+1}b$ and (ii) ($\exists c$ such that $aR_{k+1}c$ and $bR_{k+1}c$) implies ($\exists c'$ such that aR_kc' and bR_kc').*

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[2] $\forall a$, $\exists b$ such that aQb.

[3] aQb and aQc imply bQc or cQb.

[4] If $\exists a' \in W$ such that $a'Qb$, then aQc and bQc imply aQb or bQa.

[5] aR_kb and aR_kc imply bR_kc.

Proposition 4 *(i) aQa iff aR_La. (ii) (aQb and not bQa) iff ($\exists k \in \{1, \dots, L\}$ such that aR_kb and not bR_ka).*

That a is not Savage-null at a can be interpreted as a being deemed subjectively possible (at some epistemic level) at any state in the same equivalence class. By Prop. 4(i), a being not Savage-null at a has two equivalent representations in terms of accessibility relations: aQa and aR_La. Likewise, $b \gg^a a$ can be interpreted as b having a higher epistemic priority than a. By Prop. 4(ii), $b \gg^a a$ have two equivalent representations: (aQb and not bQa) and ($\exists k \in \{1, \dots, L\}$ such that aR_kb and not bR_ka). Thus, both Q and $(R_1, \dots, R_L)$ capture 'subjective possibility' and 'epistemic priority' as implied by the preferences.

If conditional continuity is strengthened to *continuity* ("the Archimedean property"; cf. [8, p. 64]), then b being deemed infinitely more likely than c at a implies that c is Savage-null. Hence, $L = 1$, and by Defs. 2 and 3, $Q = R_1$. Hence, we are left with a unique serial, transitive, and euclidian accessibility relation if preferences are continuous.

4 Defining Belief Operators

We show how belief operators can be defined using the accessibility relation of epistemic priority, Q, having the properties of Prop. 2.[6]

The set of states that are deemed subjectively possible at a is given by

$$\kappa^a = \{b \in \tau^a | \exists c \text{ such that } cQb\} = \{b \in \tau^a | bQb\},$$

where $\kappa^a \neq \emptyset$ since Q is serial, and where the last equality follows since bQb if cQb.

Definition 4 At a the decision maker *certainly believes* A if $a \in \mathrm{K}A$, where $\mathrm{K}A := \{a \in W | \kappa^a \subseteq A\}$.

Hence, at a an event A is certainly believed if the complement is deemed subjectively impossible at a. This corresponds to what Morris [20] calls 'Savage-belief'.

'Conditional belief' is defined conditionally on sets that are subjectively possible at any state; i.e., sets in the following collection:

$$\Phi := \{\phi \in 2^W \backslash \{\emptyset\} | \, \forall a \in W, \, \kappa^a \cap \phi \neq \emptyset\}.$$

[6] In the full version of this paper we show how equivalence classes can be derived from Q with the properties of Prop. 2, implying that Q with such properties suffices for defining the belief operators.

In particular, $W \in \Phi$ and, $\forall \phi \in \Phi$, $\emptyset \neq \phi \subseteq W$.

Since every $\phi \in \Phi$ is subjectively possible at any state, it follows that, $\forall \phi \in \Phi$,

$$\beta^a(\phi) := \{b \in \tau^a \cap \phi | \forall c \in \tau^a \cap \phi, cQb\} \neq \emptyset .$$

Definition 5 At a the decision maker *believes A conditional* on ϕ if $a \in \mathrm{B}(\phi)A$, where $\mathrm{B}(\phi)A := \{a \in W | \beta^a(\phi) \subseteq A\}$.

Hence, at a an event A is believed conditional on ϕ if A contains any state in $\tau^a \cap \phi$ with at least as high epistemic priority as any other state in $\tau^a \cap \phi$. This way of defining conditional belief is in the tradition of, e.g., Grove [16], Katsuno & Mendelzon [17], Boutilier [9], and Lamerre & Shoham [19].

Let Φ^A be the collection of subjectively possible events ϕ having the property that A is subjectively possible conditional on ϕ whenever A is subjectively possible:

$$\Phi^A := \{\phi \in 2^W \backslash \{\emptyset\} | \forall a \in W, \kappa^a \cap \phi \neq \emptyset, \text{ and } A \cap \kappa^a \cap \phi \neq \emptyset \text{ if } A \cap \kappa^a \neq \emptyset\} .$$

Note that Φ^A is a subset of Φ that satisfies $W \in \Phi^A$; hence, $\emptyset \neq \Phi^A \subseteq \Phi$.

Definition 6 At a the decision maker *fully believes* A if $a \in \mathrm{B}^0 A$, where $\mathrm{B}^0 A := \cap_{\phi \in \Phi^A} \mathrm{B}(\phi)A$.

Hence, at a an event A is fully believed if A is robustly believed in the following sense: A is believed conditional on any event ϕ that does not make A subjectively impossible. Indeed, B^0 corresponds to what Stalnaker [25] calls 'absolutely robust belief'. The relation between this belief operator and the operators 'assumption' and 'strong belief', introduced by Brandenburger & Keisler [13] and Battigalli & Siniscalchi [6] respectively, will be discussed in Sects. 7 and 8.

5 Characterizing Belief Operators

We now show how the certain, conditional, and full belief operators can be characterized by means the vector of nested accessibility relations $(R_1, \ldots, R_L)$ having the properties of Prop. 3 and being related to Q as in Prop. 4.[7]

Proposition 5 $\mathrm{K}A = \{a \in W | \rho_L^a \subseteq A\}$.

[7]In the full version of this paper we first derive $(R_1, \ldots, R_L)$ from Q and then show how $(R_1, \ldots, R_L)$ characterizes the belief operators.

Proposition 6 $\forall\phi \in \Phi$, $\mathrm{B}(\phi)A = \{a \in W | \exists k \in \{1,\dots,L\}$ such that $\emptyset \neq \rho_k^a \cap \phi \subseteq A\}$.

Proposition 7 $\mathrm{B}^0 A = \{a \in W | \exists k \in \{1,....,L\}$ such that $\rho_k^a = A \cap \kappa^a\}$.

In combination with Prop. 4(ii) Prop. 7 means that A is fully believed iff any subjectively possible state in A has higher epistemic priority than any state in the same equivalence class outside A.

6 Properties of Belief Operators

In the full version of this paper we show that certain belief and conditional belief are KD45 operators, and that conditional belief satisfies the usual properties for belief revision as given by Stalnaker [25] (see also Alchourrón et al. [1]). The full belief operator is, however, not as well behaved.

Proposition 8 $\mathrm{K}A \subseteq \mathrm{B}^0 A \subseteq \mathrm{B}(W)A$.

Proposition 9 *The following properties hold: (i)* $\mathrm{B}^0 A \cap \mathrm{B}^0 A' \subseteq \mathrm{B}^0(A \cap A')$, *(ii)* $\mathrm{B}^0 A \subseteq \mathrm{K}\mathrm{B}^0 A$, *and (iii)* $\neg\mathrm{B}^0 A \subseteq \mathrm{K}(\neg\mathrm{B}^0 A)$.

Note that $\mathrm{B}^0\emptyset = \emptyset$, $\mathrm{B}^0 W = W$, $\mathrm{B}^0 A \subseteq \mathrm{B}^0\mathrm{B}^0 A$ and $\neg\mathrm{B}^0 A \subseteq \mathrm{B}^0(\neg\mathrm{B}^0 A)$ follow from Props. 8 and 9 since $B(W)\emptyset = \emptyset$ and $KW = W$. However, even though the operator B^0 satisfies $B^0 A \subseteq \neg B^0 \neg A$ as well as positive and negative introspection, it does not satisfy monotonicity since $A \subseteq A'$ does not imply $B^0 A \subseteq B^0 A'$. To see this let $\rho_1^a = \{a\}$ and $\rho_2^a = \kappa^a = \{a, b, c\}$ for some $a \in W$. Now let $A = \{a\}$ and $A' = \{a, b\}$. Clearly, $A \subseteq A'$, and since $\rho_1^a = A \cap \kappa^a$ we have $a \in B^0 A$. However, since neither $\rho_1^a = A' \cap \kappa^a$ nor $\rho_2^a = A' \cap \kappa^a$, $a \notin B^0 A'$. Hence, though full belief is thus bounded by two KD45 operators, full belief is not itself a KD45 operator.

7 Epistemic analysis of strategic form games

Let S_i denote player i's finite set of *pure strategies*, and let $z : S \to Z$ map strategy vectors into outcomes, where $S = S_1 \times S_2$ is the set of strategy vectors and Z is the finite set of outcomes. Then (S_1, S_2, z) is a finite *strategic* two-person *game form*.

For each player i, any of i's strategies is an act from strategy choices of his opponent j to outcomes. The uncertainty faced by a player i in a strategic game form concerns j's strategy choice, j's preferences over acts from i's strategy choices to outcomes, and so on (see Tan & Werlang [26]). A type of a player i corresponds to preferences over acts from j's strategy choices, preferences over acts from j's preferences over acts from i's strategy choices, and so on.

By adding subscript i to the framework introduced in Sect. 2, the finite set of states (or possible worlds) for player i, W_i, can be interpreted as

$$W_i = T_i \times S_j \times T_j\,,$$

where, for each $a \in W_i$, the set of states in the same equivalence class as a equals

$$\tau_i^a = \{t_i^a\} \times S_j \times T_j\,,$$

with t_i^a denoting the type of i determined by the state a. The property that all three belief operators satisfy positive and negative introspection, corresponds to the property that at any state $a \in W_i$ (and any conditioning event), player i certainly believes/conditionally believes/fully believes that he is of type t_i^a.

Definition 7 An *interactive preference structure* for the strategic game form (S_1, S_2, z) is a structure

$$(S_1, S_2, \{\succeq_1^a \mid a \in T_1 \times S_2 \times T_2\}, \{\succeq_2^a \mid a \in T_2 \times S_1 \times T_1\})\,,$$

where, for each i, $\{\succeq_i^a \mid a \in T_i \times S_j \times T_j\}$ satisfies Def. 1 with, for all $a \in T_i \times S_j \times T_j$, $\tau_i^a = \{t_i^a\} \times S_j \times T_j$.

Asheim & Dufwenberg [4] employ an interactive preference structure like the one described in Def. 7. They say that an event A is fully believed at a if the preferences at a are admissible on the set of states in A that are deemed subjective possible at a. It follows from Prop. 7 that this corresponds to full belief as defined in Def. 6.

Refer again to the decision maker without subscript i. Brandenburger & Keisler [13] consider an interactive preference structure where, for any $a \in W$, $\succeq^a$ is assumed to satisfy completeness as well as *partitional continuity* (in the sense of [8, Axiom 4″]). Then Blume et al. [8, Thm. 5.3] implies that $\succeq^a$ is represented by u^a and a *lexicographic conditional probability system* (LCPS) — i.e., a hierarchy of subjective probability distributions with non-overlapping supports where the support of the k-level probability distribution p_k^a equals π_k^a, and where $\{\pi_1^a, \dots, \pi_{L^a}^a\}$ is a partition of κ^a (cf. [8, Def. 5.2]). Brandenburger & Keisler [13, Appendix B] employ an LCPS to define the preferences in their setting where the set of states is infinite, and they [13, Def. B1] introduce the following belief operator.

Definition 8 (Brandenburger & Keisler [13]) At a the decision maker *assumes* A if $\succeq^a_A$ is nontrivial and $\mathbf{x} \succ^a_A \mathbf{y}$ implies $\mathbf{x} \succ^a \mathbf{y}$.

Proposition 10 *Assume that $\succeq^a$ is complete and satisfies partitional continuity. Then A is assumed at a iff $a \in B^0 A$.*

Proposition 10 shows that the 'assumption' operator coincides with full belief (and thus with Stalnaker's [25] 'absolutely robust belief') under completeness and partitional continuity. However, if partitional continuity is weakened to conditional continuity, then this equivalence is not obtained. In the full version of this paper, we use the example of Blume et al. [8, Sect. 5] to show this. Brandenburger & Keisler [13] do not indicate that their definition – as stated in Def. 8 – should be used outside the realm of partionally continuous preferences. Hence, our definition of full belief – combined with its characterization in Prop. 7 – yields a preference-based generalization of Brandenburger & Keisler's [13] operator to preferences that need only satisfy the properties of Def. 1.

8 Epistemic analysis of extensive form games

Cf. Osborne & Rubinstein [22, Ch. 6] for the details of a finite extensive two-person game form of almost perfect information with H denoting the set of *subgames* and Z denoting the set of outcomes. A pure strategy s_i assigns a feasible action to any $h \in H$. Denote by S_i player i's finite set of pure strategies, and write $S := S_1 \times S_2$. Let $z : S \to Z$ map strategy vectors into outcomes. Then (S_1, S_2, z) is a finite strategic two-person game form. For any $h \in H$, let $S(h) = S_1(h) \times S_2(h)$ denote the set of strategy vectors that are *consistent* with h being reached. Note that $S(\emptyset) = S$.

Thus, also in a multi-stage game, the set of states (or possible worlds) for each player i, W_i, can be interpreted as $W_i = T_i \times S_j \times T_j$ with, for all $a \in W_i$, $\tau^a_i = \{t^a_i\} \times S_j \times T_j$. Let Φ^H_i denote the collection of subsets that correspond to subgames:

$$\Phi^H_i := \{\phi \in 2^{W_i} \backslash \{\emptyset\} \,|\, \exists h \in H \text{ s.t. } \phi = T_i \times S_j(h) \times T_j\}\,.$$

Assume that any subgame is subjectively possible at any state: For all $a \in W_i$ and $h \in H$, $\kappa^a_i \cap (T_i \times S_j(h) \times T_j) \neq \emptyset$, implying that $\Phi^H_i \subseteq \Phi_i$.

A system of conditional preferences in an extensive form game need not consist of conditional binary relations derived from a single binary relation over acts on W_i, as implied by Def. 7. E.g., sequential equilibrium requires a system of conditional preferences that cannot be made up of conditional binary relations derived from a single binary relation over acts on W_i (cf. Asheim & Perea [5, Sect. 2]). To show how belief operators derived from a system of conditional preferences relate to the belief operators defined in Sect. 4, a system of conditional preferences must be made isomorphic to preferences consistent with Def. 1.

Refer again to a decision maker without the subscript i. As before, for any $a \in A$, let τ^a denote the equivalence class to which a belongs, let κ^a denote the set of subjectively possible states at a, and let Φ denote the collection of sets that are subjectively possible at any state. For each $a \in W$, consider a *system of conditional preferences*, $\{\succeq^a_\phi \mid \phi \in \Phi\}$, in the following sense: For any $\phi \in \Phi$, the preferences of the decision maker conditional on ϕ is a binary relation $\succeq^a_\phi$ on the set of acts on ϕ; i.e., on the set of functions $\mathbf{x}_\phi : \phi \to \Delta(Z)$.

When one considers the above setting, Battigalli & Siniscalchi [6] and Ben-Porath [7] in effect invoke assumptions, under which a system of conditional preferences, $\{\succeq^a_\phi \mid \phi \in \Phi\}$, is isomorphic to a vNM utility function u^a and an LCPS, $(p^a_1, \dots, p^a_{L^a})$ — where, for each $k \in \{1, \dots, L^a\}$, $\text{supp}p^a_k = \pi^a_k$ and $\{\pi^a_1, \dots, \pi^a_{L^a}\}$ is a partition of κ^a — in the following sense (cf. Blume et al. [8, Sect. 5]): For any $\phi \in \Phi$, $\succeq^a_\phi$ is represented by u^a and the probability distribution p^a_ϕ, where is p^a_ϕ is the conditional of p^a_ℓ on ϕ, with $\ell := \min\{k \in \{1, \dots, L^a\} | \, \pi^a_k \cap \phi \neq \emptyset\}$. The system of probability distributions, $\{p^a_\phi | \, \phi \in \Phi\}$, is called a *conditional probability system* (cf. Myerson [21]). Battigalli & Siniscalchi [6] and Ben-Porath [7] employ a CPS to define the system of conditional preferences.

Battigalli & Siniscalchi [6] and Ben-Porath [7] define conditional belief with probability 1: At a the decision maker believes A conditional on ϕ if $\text{supp}p^a_\phi \subseteq A$. In the full version, we show that this conditional belief operator coincides with the $B(\phi)$ operator of the present paper. Therefore, we can define Battigalli & Siniscalchi's [6] 'strong belief' operator as follows, where $\Phi^H \cap \Phi^A$ is the collection of subgames ϕ having the property that A is subjectively possible conditional on ϕ whenever A is subjectively possible.

Definition 9 (Battigalli & Siniscalchi [6]) At a the decision maker *strongly believes* A if $a \in \bigcap_{\Phi^H \cap \Phi^A} B(\phi)$.

Hence, at a an event A is strongly believed if A is robustly believed in the following sense: A is believed conditional on any subgame ϕ that does not make A subjectively impossible. Since $\Phi^A \supseteq \Phi^H \cap \Phi^A \supseteq \{W\}$, we obtain the following result.

Proposition 11 *If $a \in B^0(A)$, then A is strongly believed at a. If A is strongly believed at a, then $a \in B(W)A$.*

The 'strong belief' operator shares the properties of full belief: It satisfies the properties of Prop. 9, but is not monotonic.

9 Concluding Remarks

We have presented a model with (i) a serial, transitive, forwardly linear and quasi-backwardly linear epistemic priority relation Q, and, equivalently, (ii) a vector of nested, serial, transitive and euclidean accessibility relations $(R_1, \ldots, R_L)$. These give two equivalent representations of the notions of 'subjective possibility' and 'epistemic priority', on which the concepts of certain, conditional, and full belief are based. Both Q and $(R_1, \ldots, R_L)$ can be derived from preferences that need not be complete and thus representable by subjective probabilities.

This model does not require that the epistemic priority relation is reflexive. The decision maker may be subjectively unable to distinguish between two objectively possible states, while deeming (at the lowest epistemic level) that one is subjectively possible and the other not. Because Q lacks reflexivity, not even the certain belief operator obeys the truth axiom; thus, we allow that the decision maker holds the true state as subjectively impossible (even at the lowest epistemic level).

The distinction between subjectively possible and subjectively impossible events can be illustrated within an interactive preference structure for the strategic game form of a multi-stage game (cf. Def. 7 and Sect. 8). If each player considers any opponent strategy to be subjectively possible, then any $\phi \in \Phi^H$ (the collection of subsets of states that correspond to subgames) will be subjectively possible, as well as potentially observable (cf. Brandenburger [11]). The player can still deem it subjectively impossible that the opponent holds particular preferences, as the preferences of the opponent are not directly observable. Brandenburger & Keisler [12] show that there need not exist a preference-complete interactive preference structure when preferences are not representable by subjective probabilities. This result makes models that do not require a decision maker to hold all objectively possible opponent preferences as subjectively possible particularly relevant in game-theoretic applications.

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