Boolean Games

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Abstract

In this paper Boolean games are introduced as a class of two-player zero-sum games along with a number of operations on them. We argue that Boolean games can be interpreted as modelling the information structures of two-person zero-sum games. As such they comprise games of imperfect information. The algebra of Boolean games *modulo* strategic equivalence is then proven to be isomorphic to the Lindenbaum algebra of Classical Propositional Logic. A neat match between the game-theoretical notion of a winning strategy and a logical counterpart, however, calls for a refinement of the notion of validity. Pursuing this issue we finally obtain a logical characterization of determinacy for Boolean games.
Keywords: Zero-sum Games, Boolean Algebra, Classical Propositional Logic, Determinacy.

1 Introduction

One of the early issues in game theory was under which operations extensive games remain strategically equivalent. This question has a distinctly algebraic ring and gives rise to others, such as "what are viable operations on games?", "what is a feasible notion of strategic equivalence?" and, given satisfactory answers to these, "what kind of algebra do games constitute?".

In this paper, these algebraic issues are apportioned a central place as they mediate a logical approach to some of the qualitative aspects of game theory. We will subsequently:

(1) define inductively a class of two-person fully competitive games, which comprises both games of perfect and imperfect information;

(2) introduce a viable notion of strategic equivalence;

(3) prove the games constitute a Boolean algebra *modulo* this notion of strategic equivalence;

(4) formulate a sound and complete calculus for winning strategies;

(5) establish a correspondence with an appropriate classical propositional language and prove the Lindenbaum algebra of the respective logic to be isomorphic to the algebra of games;

(6) generalize the logical notions of validity and satisfiability and show these refined concepts to correspond to game-theoretical ones;

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(7) give a *logical* characterization of *determined* games which also applies to games of imperfect information.

This opens up a line of research in which game-theoretical issues are attended to with solely logical means. Our concerns are thus with the logical foundations of game theory and strategic reasoning rather than with the game-theoretical underpinnings of logic. It is in this respect that our approach should be distinguished from game-theoretical semantics (cf., Hintikka [1973], Hintikka and Sandu [1997]), the employment of games in model theory (cf., Hodges [1993]), and logic games (cf., Lorenzen and Lorenz [1978]). It is far more reminiscent of the recent work by van Benthem (cf., van Benthem[1998], [1999], [2000]), although the formal development of our framework is different from any of the above.

Both two-person zero-sum games and classical propositional logic have been studied extensively. As such, the merits of this paper should be sought mainly in the light it sheds on how logic, algebra and game-theory are intertwined. Moreover, the authors believe that the present analysis provides the necessary foundations for a more extensive framework facilitating a logical theory of multi-player games.

2 Games

The games we consider have $\{0,1\}$ as the set of players. With each player $i \in \{0,1\}$ we associate a set of atomic strategies $A_i = \{a_0^i, \ldots, a_n^i, \ldots\}$. Let $A_0 \cap A_1 = \varnothing$ and have A denote $A_0 \cup A_1$. Relative to A we define inductively the set of games $\mathfrak{G}_A$. At the lowest level we distinguish two *atomic* games, $\mathbf{0}$ and $\mathbf{1}$. In the former, intuitively, player 1 loses and player 0 wins, no matter what. The constant outcome of this game we denote by the player that wins, i.e., 0. In contrast, in $\mathbf{1}$ victory is player 1's and player 0 will have to content himself with a defeat. The outcome of this game we denote by 1.

A molecular game $\alpha(\mathfrak{g}_0, \mathfrak{g}_1)$ offers either player 0 or player 1, depending on whether $\alpha \in A_0$ or $\alpha \in A_1$, the choice between performing α or refraining from this course of action. The outcome of this game will be $\mathfrak{g}_0$ or $\mathfrak{g}_1$, respectively.

Formally, games are introduced as sequences of atomic strategies and atomic games:

Definition 2.1 (*Games*) Let A be a set of atomic strategies and define the set of games $\mathfrak{G}$ as the smallest set satisfying both:

(*i*) $\{\mathbf{0},\mathbf{1}\} \subseteq \mathfrak{G}_A$

(*ii*) $\mathfrak{g}_0, \mathfrak{g}_1 \in \mathfrak{G}_A \;\&\; \alpha \in A \Longrightarrow (\alpha, \mathfrak{g}_0, \mathfrak{g}_1) \in \mathfrak{G}_A$

The game $(\alpha, \mathfrak{g}_0, \mathfrak{g}_1)$ will in the sequel generally be denoted by $\alpha(\mathfrak{g}_0, \mathfrak{g}_1)$. The subscript in $\mathfrak{G}_A$ will usually be omitted.

Note that, although in each game a player has a binary choice, we do not assume that the players alternate. Consequently, n-ary choices for a player can be accounted for by having a player make $n-1$ binary choices in a row.

Molecular games are thought of as being played iteratively. First, one player plays a game $\alpha(\mathfrak{g}_0, \mathfrak{g}_1)$ by performing or refraining from doing α and her choice determines which of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ is played subsequently. This sequentiality can be made explicit by representing a game $\mathfrak{g} \in \mathfrak{G}$ as a tree. Interpreting movement to the left at a vertex labelled with α as performing α and going to the right as refraining from acting thus, the tree form of $\mathfrak{g} = a^1(a^0(\mathbf{1}, b^1(\mathbf{1},\mathbf{0})), b^0(\mathbf{0},\mathbf{1}))$ could be depicted as in figure 2.1:

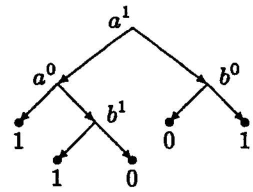

Figure 2.1

3 Operations on Games

We now introduce a total of four operations on games by means of which games can be constructed from smaller ones.

Definition 3.1 Let $\mathfrak{g}$ and $\mathfrak{h}$ be games. Define $+$, $\cdot$, $\overline{}$ and $\oplus$ such that:

(1) $\mathbf{0} + \mathfrak{h} := \mathfrak{h}$
$\mathbf{1} + \mathfrak{h} := \mathbf{1}$
$\alpha(\mathfrak{g}_0, \mathfrak{g}_1) + \mathfrak{h} := \alpha(\mathfrak{g}_0 + \mathfrak{h}, \mathfrak{g}_1 + \mathfrak{h})$

(2) $\mathbf{0} \cdot \mathfrak{h} := \mathbf{0}$
$\mathbf{1} \cdot \mathfrak{h} := \mathfrak{h}$
$\alpha(\mathfrak{g}_0, \mathfrak{g}_1) \cdot \mathfrak{h} := \alpha(\mathfrak{g}_0 \cdot \mathfrak{h}, \mathfrak{g}_1 \cdot \mathfrak{h})$

(3) $\overline{\mathbf{0}} := \mathbf{1}$
$\overline{\mathbf{1}} := \mathbf{0}$
$\overline{(\alpha(\mathfrak{g}_0, \mathfrak{g}_1))} := \alpha(\overline{\mathfrak{g}}_0, \overline{\mathfrak{g}}_1)$

(4) $\oplus(\mathbf{0}, \mathfrak{h}, \mathfrak{k}) := \mathfrak{k}$
$\oplus(\mathbf{1}, \mathfrak{h}, \mathfrak{k}) := \mathfrak{h}$
$\oplus(\alpha(\mathfrak{g}_0, \mathfrak{g}_1), \mathfrak{h}, \mathfrak{k}) := \alpha(\oplus(\mathfrak{g}_0, \mathfrak{h}, \mathfrak{k}), \oplus(\mathfrak{g}_1, \mathfrak{h}, \mathfrak{k}))$

Intuitively, the sum of two games, $\mathfrak{g} + \mathfrak{h}$, is the result of replacing *any* occurrence of the game $\mathbf{0}$ in $\mathfrak{g}$ by $\mathfrak{h}$. Conceiving games as trees, $+$ makes that the root of $\mathfrak{h}$ will be attached to any leaf node of $\mathfrak{g}$ labelled with 0 (cf. figure 3.1).

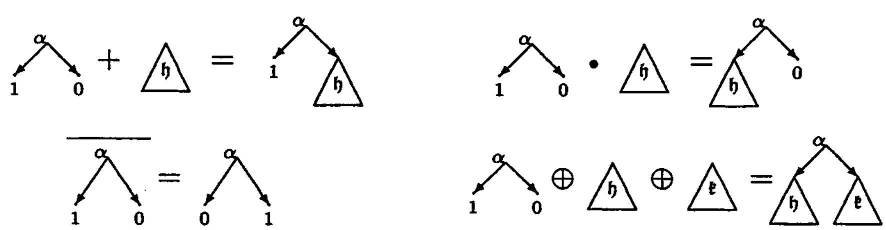

Figure 3.1

Taking the product of two games ($\mathfrak{g} \cdot \mathfrak{h}$) is similar to adding two games, be it that through $\cdot$ it is every occurrence of $\mathbf{1}$ that is being replaced by $\mathfrak{h}$. The complement of a game $\mathfrak{g}$, $\overline{\mathfrak{g}}$, differs from $\mathfrak{g}$ in that all occurrences of $\mathbf{1}$ and $\mathbf{0}$ are interchanged. The operation $\oplus$ comes down to simultaneously adding one game and multiplying it with another. Hence, $\oplus(\mathfrak{g}, \mathfrak{h}, \mathfrak{k})$ is the game that is like $\mathfrak{g}$ except that each occurrence of $\mathbf{1}$ is replaced by an occurrence of $\mathfrak{h}$ and every occurrence of $\mathbf{0}$ by one of $\mathfrak{k}$. It be noted that $\{\alpha(\mathbf{1}, \mathbf{0}) \mid \alpha \in A\} \cup \{\mathbf{0}, \mathbf{1}\}$ is a set of generators in the algebra $(\mathfrak{G}_A, +, \cdot, \overline{}, \oplus)$. No subset of $\{+, \cdot, \overline{}\}$ is functionally complete in this respect, although $\oplus$ on its own is.

4 Strategies & Strategy Profiles

So far, only a few, informal, words have been spent on what *strategies* the players can adopt in playing the games introduced in Definition 2.1 and different options are still open in this respect. Here, we take the powerset of A_i, *c.q.* the set of characteristic functions on A, 2^{A_i}, as the set of strategies for player i ($i \in \{0,1\}$). A *strategy profile* is the set-theoretic union of a strategy s of player 0 and a strategy s' of player 1. Intuitively, when a player i adopts a strategy $s \in S_i$, i will perform α in a game $\alpha(\mathfrak{g}_0, \mathfrak{g}_1)$ if $\alpha \in s$, and refrain from acting thus otherwise. This conception of a strategy has some interesting consequences.

Definition 4.1 (*Strategies*) Define for each $i \in \{0,1\}$ a set of strategies S_i as well as a set of strategy profiles Σ as:

$$\begin{aligned} S_i &:= 2^{A_i} \\ \Sigma &:= \{s \cup s' \mid s \in S_0 \ \&\ s' \in S_1\} \qquad (= 2^A) \end{aligned}$$

A strategy profile $\sigma \in \Sigma$ can be regarded as determining a unique outcome value in $\{0,1\}$ for each game. For $\mathbf{0}$ and $\mathbf{1}$ this outcome will invariably be 0 and 1, respectively. The outcome of a molecular game given a strategy profile σ will, strictly speaking, again be a game. However, assuming that players do not revise their strategies during play, a unique value in $\{0,1\}$ can be associated with each game $\mathfrak{g} \in \mathfrak{G}$ and each $\sigma \in \Sigma$ through iteratively calculating the outcomes of the games that will be outcomes if σ is played. We thus obtain the *strategic form* of a game $\mathfrak{g}$, which maps Σ onto $\{0,1\}$:

Definition 4.2 (*Strategic Form*) The strategic form of a game $\mathfrak{g}$ is a function $\mathsf{sf}(\mathfrak{g}) : \Sigma \to \{0,1\}$, defined such that for all $\sigma \in \Sigma$:

$$\begin{aligned} \mathsf{sf}(\mathbf{0})(\sigma) &:= 0 \\ \mathsf{sf}(\mathbf{1})(\sigma) &:= 1 \\ \mathsf{sf}(\alpha(\mathfrak{g}_0, \mathfrak{g}_1))(\sigma) &:= \begin{cases} \mathsf{sf}(\mathfrak{g}_0)(\sigma) & \text{if } \alpha \in \sigma \\ \mathsf{sf}(\mathfrak{g}_1)(\sigma) & \text{otherwise} \end{cases} \end{aligned}$$

We are now in a position to express the game-theoretical concept of a *winning strategy.* A strategy s is winning for a player i in a game $\mathfrak{g}$ if, no matter which strategy the other player plays, adopting s will result in a victory for i.

Definition 4.3 (*Winning Strategy*) Let $i \in \{0,1\}$. For all $\mathfrak{g} \in \mathfrak{G}$ and $s \in S_i$:
s is a *winning strategy* for i in $\mathfrak{g} :\Longleftrightarrow \forall s' \in S_{1-i} : \mathsf{sf}(\mathfrak{g})(s \cup s') = i$

If two games have the same function from Σ to $\{0,1\}$ as their strategic form, they are in an important sense equivalent. Though they may be different in many other respects, any pair of such games are quite indistinguishable from the perspective of a player that has to settle for a strategy and who is only interested in whether she wins or loses the game. These concerns give rise to the following definition of *strategic equivalence* between games:

Definition 4.4 (*Strategic Equivalence*) For all $\mathfrak{g}, \mathfrak{h} \in \mathfrak{G}$:
$\mathfrak{g} \equiv \mathfrak{h} :\Longleftrightarrow \mathsf{sf}(\mathfrak{g}) = \mathsf{sf}(\mathfrak{h})$

It is important to note that our notion of a strategy profile is global in the sense that $\mathsf{sf}(\mathfrak{g})(\sigma)$ is defined for all $\mathfrak{g} \in \mathfrak{G}$ and any $\sigma \in \Sigma$. This makes that the strategic properties of any two games, however different, can be compared in a straightforward fashion. The behavior of the strategic forms of games under the algebraic operations is summarized in the following proposition:

Proposition 4.5 For all games $\mathfrak{g}, \mathfrak{h} \in \mathfrak{G}$ and each strategy profile $\sigma \in \Sigma$:

$$\begin{array}{rcl} \mathsf{sf}(\alpha(\mathbf{1},\mathbf{0}))(\sigma) = 1 & \iff & \alpha \in \sigma \\ \mathsf{sf}(\mathfrak{g}+\mathfrak{h})(\sigma) = 1 & \iff & \mathsf{sf}(\mathfrak{g})(\sigma) = 1 \ \text{ or } \ \mathsf{sf}(\mathfrak{h})(\sigma) = 1 \\ \mathsf{sf}(\mathfrak{g}\cdot\mathfrak{h})(\sigma) = 1 & \iff & \mathsf{sf}(\mathfrak{g})(\sigma) = 1 \ \ \& \ \ \mathsf{sf}(\mathfrak{h})(\sigma) = 1 \\ \mathsf{sf}(\overline{\mathfrak{g}})(\sigma) = 1 & \iff & \mathsf{sf}(\mathfrak{g})(\sigma) = 0 \end{array}$$

We merely mention that $\oplus(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}_1) \equiv (\mathfrak{g}\cdot\mathfrak{h}_0) + (\overline{\mathfrak{g}}\cdot\mathfrak{h}_1)$. Hence, from a strategic perspective, $\oplus$ can be dispensed with.

A more important consequence of both Proposition 4.5 and the choice of strategy profiles as subsets of A is that the set of games can be proved to constitute a Boolean algebra *modulo* strategic equivalence. Strategic equivalence partitions $\mathfrak{G}$. Let $[\mathfrak{g}]$ denote $\{\mathfrak{h} \mid \mathfrak{g} \equiv \mathfrak{h}\}$ and set $\mathfrak{G}_{\equiv} := \{[\mathfrak{g}] \mid \mathfrak{g} \in \mathfrak{G}\}$. The operations $+$, $\cdot$ and $^{-}$ can be raised to operations on equivalence classes of games, $+$, $\bullet$ and $^{-}$, in such a way that $[\mathfrak{g}] + [\mathfrak{h}] = [\mathfrak{g}+\mathfrak{h}]$, $[\mathfrak{g}] \bullet [\mathfrak{h}] = [\mathfrak{g}\cdot\mathfrak{h}]$, and $\overline{[\mathfrak{g}]} = [\overline{\mathfrak{g}}]$. We now obtain the following theorem.

Theorem 4.6 $(\mathfrak{G}_{\equiv}, +, \bullet, ^{-}, [\mathbf{0}], [\mathbf{1}])$ is a Boolean algebra.

5 A Calculus for Winning Strategies

In this section we introduce a sound and complete system to derive winning strategies by mere symbolic manipulation. To this end we first introduce the notion of a *partial* strategy profile.

Formally, a partial strategy profile is a pair of disjoint strategy profiles $(\sigma, \tau) \in \Sigma \times \Sigma$. Intuitively, the first entry denotes the atomic strategies that are to be played, the second entry the atomic strategies that are to be refrained from. The partiality of a partial strategy profile lies in the fact that it does not in general determine for each atomic strategy whether it should be played or rather not.

Definition 5.1 (*Partial Strategy Profiles*) Define the set of *partial strategy profiles* Σ^{π} as:

$$\Sigma^{\pi} := \{(\sigma,\tau) \in \Sigma \times \Sigma \mid \sigma \cap \tau = \varnothing\}$$

Partial strategies can be ordered in accordance with their measure of specification. Let $\sqsubseteq$ be such that for all $(\sigma,\tau), (\sigma',\tau') \in \Sigma^{\pi}$: $(\sigma,\tau) \sqsubseteq (\sigma',\tau') :\iff \sigma \subseteq \sigma' \ \& \ \tau \subseteq \tau'$. $(\Sigma^{\pi}, \sqsubseteq)$ is then a partially ordered set in which $(\varnothing, \varnothing)$ is the least element in the order, and, letting $\overline{\sigma}$ denote $A \backslash \sigma$, each $(\sigma, \overline{\sigma})$ a maximal element. If $(\sigma,\tau) \sqsubseteq (\sigma',\tau')$, we say that (σ',τ') *extends* (σ,τ). The definition of the strategic form of a game can be conservatively extended to a *partial* function from partial strategy profiles to $\{0,1\}$. The *partial strategic form* of a game $\mathfrak{g}$, $\mathsf{pf}(\mathfrak{g})$, is defined for a partial strategy profile (σ,τ) if it determines the same value for all maximal extensions of (σ,τ) and is undefined otherwise.

Definition 5.2 (*Partial Strategic Form*) The *partial strategic form* of a game $\mathfrak{g}$ is a partial function $\mathsf{pf}(\mathfrak{g}) : \Sigma^{\pi} \to \{0,1\}$ such that for $i \in \{0,1\}$:

$$\mathsf{pf}(\mathfrak{g})((\sigma,\tau)) := \begin{cases} i & \text{if } \forall v \in \Sigma : (\sigma,\tau) \sqsubseteq (v,\overline{v}) \Longrightarrow \mathsf{sf}(\mathfrak{g})(v) = i \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that the following proposition is an almost immediate consequence of the definitions:

Proposition 5.3 Let $i \in \{0,1\}$. For all $\mathfrak{g} \in \mathfrak{G}$, $\sigma \in \Sigma$ and $s \in S_i$:

$$\begin{array}{lrcl} (i) & \mathsf{pf}(\mathfrak{g})((\sigma,\overline{\sigma})) = i & \iff & \mathsf{sf}(\mathfrak{g})(\sigma) = i \\ (ii) & \mathsf{pf}(\mathfrak{g})((s, S_i \backslash s)) = i & \iff & s \text{ is a winning strategy for player } i \end{array}$$

We are now in a position to present the calculus and prove its soundness and completeness. Intuitively, $(\sigma,\tau)\,\|\!/_i\,\mathfrak{g}$ signifies that the partial strategy profile (σ,τ) guarantees player i a win in game $\mathfrak{g}$.

Definition 5.4 ($\|\!/_i$) Let $i \in \{0,1\}$. Define for all $(\sigma,\tau),(\sigma',\tau') \in \Sigma^\pi$:

$$A0^0: \quad (\varnothing,\varnothing)\,\|\!/_0\,\mathbf{0} \qquad\qquad A0^1: \quad (\varnothing,\varnothing)\,\|\!/_1\,\mathbf{1}$$

$$R1^i: \frac{(\sigma,\tau)\,\|\!/_i\,\mathfrak{g}_0 \qquad \alpha\notin\tau}{(\sigma\cup\{\alpha\},\tau)\,\|\!/_i\,\alpha(\mathfrak{g}_0,\mathfrak{g}_1)} \qquad\qquad R2^i: \frac{(\sigma,\tau)\,\|\!/_i\,\mathfrak{g}_1 \qquad \alpha\notin\sigma}{(\sigma,\tau\cup\{\alpha\})\,\|\!/_i\,\alpha(\mathfrak{g}_0,\mathfrak{g}_1)}$$

$$R3^i: \frac{(\sigma\cup\{\beta\},\tau)\,\|\!/_i\,\mathfrak{g} \qquad (\sigma,\tau\cup\{\beta\})\,\|\!/_i\,\mathfrak{g}}{(\sigma,\tau)\,\|\!/_i\,\mathfrak{g}}$$

$$R4^i: \frac{(\sigma,\tau)\,\|\!/_i\,\mathfrak{g} \qquad \sigma\cap\tau' = \sigma'\cap\tau = \sigma'\cap\tau' = \varnothing}{(\sigma\cup\sigma',\tau\cup\tau')\,\|\!/_i\,\mathfrak{g}}$$

The derivability relation $\|\!/_i \subseteq \Sigma^\pi \times \mathfrak{G}$ is then given by closing the set of instances of axiom $A0^i$ under $R1^i$–$R4^i$.

This system turns out to be both sound and complete with respect to the extended notion of the strategic form of a game.

Theorem 5.5 (*Soundness & Completeness*) Let $i \in \{0,1\}$. For all games $\mathfrak{g} \in \mathfrak{G}$, $(\sigma,\tau) \in \Sigma^\pi$:
$(\sigma,\tau)\,\|\!/_i\,\mathfrak{g} \iff \mathsf{pf}(\mathfrak{g})((\sigma,\tau)) = i$

Proof: Soundness is proved by an induction on the length of the derivation, whereas completeness can be proved by an induction on the complexity of $\mathfrak{g}$. $\dashv$

As a corollary we obtain the following, which establishes the system as a calculus for winning strategies:

Corollary 5.6 Let $i \in \{0,1\}$. For all $\mathfrak{g} \in \mathfrak{G}$, $s \in S_i$:
$(s,\overline{s})\,\|\!/_i\,\mathfrak{g} \iff s$ is a winning strategy for i in $\mathfrak{g}$

Another issue is how the derivability relations $\|\!/_i$ behave with respect to the game operators $+, \cdot, \overline{}$. Derived rules for these can easily be obtained in the calculus, e.g., the following. Assume $\sigma\cap\tau' = \sigma'\cap\tau = \sigma'\cap\tau' = \varnothing$.

$$A1^0: \quad (\varnothing,\{\alpha\})\,\|\!/_0\,\alpha(\mathbf{1},\mathbf{0}) \qquad\qquad A1^1: \quad (\{\alpha\},\varnothing)\,\|\!/_1\,\alpha(\mathbf{1},\mathbf{0})$$

$$R5^1: \frac{(\sigma,\tau)\,\|\!/_1\,\mathfrak{g}}{(\sigma,\tau)\,\|\!/_1\,\mathfrak{g}+\mathfrak{h}} \qquad\qquad R6^0: \frac{(\sigma,\tau)\,\|\!/_0\,\mathfrak{h}}{(\sigma,\tau)\,\|\!/_0\,\mathfrak{g}\cdot\mathfrak{h}}$$

$$R7^1: \frac{(\sigma,\tau)\,\|\!/_1\,\mathfrak{g} \qquad (\sigma',\tau')\,\|\!/_1\,\mathfrak{h}}{(\sigma\cup\sigma',\tau\cup\tau')\,\|\!/_1\,\mathfrak{g}\cdot\mathfrak{h}} \qquad\qquad R8^i: \frac{(\sigma,\tau)\,\|\!/_{1-i}\,\mathfrak{g}}{(\sigma,\tau)\,\|\!/_i\,\overline{\mathfrak{g}}}$$

6 Perfect & Imperfect Information

When setting out to play a Boolean game $\mathfrak{g}$, a player i has to decide for each atomic strategy $a^i \in A_i$ whether to admit it to her strategy or not. Moreover, she has to stick to this choice throughout the game, even though a^i may occur more than once in $\mathfrak{g}$. Within the Boolean framework there simply is no strategy $s \in S_i$ such that i's adopting it makes that a (sub-)game $a^i(\mathfrak{g}_0, \mathfrak{g}_1)$ is continued with $\mathfrak{g}_0$ and another (sub-)game $a^i(\mathfrak{h}_0, \mathfrak{h}_1)$ with $\mathfrak{h}_1$. So, if atomic strategies are interpreted as (choices between) actual actions, it could be argued that Boolean games comprise rather a restricted class of games, with only a limited number of real-world examples.

Boolean games, however, also allow for an alternative and more comprehensive interpretation. Rather than denoting choices between two actual courses of action, the atomic strategies could be taken as labelling the sets of game-states the players cannot epistemically tell apart. Under this interpretation a player i is supposed to be incapable of distinguishing the state in which she has to decide on a course of action in $a^i(\mathfrak{g}_0, \mathfrak{g}_1)$ from the one she is in when having to move in $b^i(\mathfrak{h}_0, \mathfrak{h}_1)$ if and only if $a^i = b^i$. We argue that, conceived thus, Boolean games provide a framework in which the information structures of finite fully competitive two-person zero-sum games can be modelled.

Generally speaking, in games of imperfect information situations may arise in which some player is unable to distinguish the state she is actually in from a state that she could have been in had the game been played differently. Players, caught as they are in their epistemic state, should settle on strategies that prescribe the same course of action in any two states that they cannot tell apart. In this way, imperfect information restricts the number of strategies available to a player. This complies with the interpretation of Boolean games in which atomic strategies are taken as labels of information states and strategies as sets of atomic strategies. Any two subgames $\alpha(\mathfrak{g}_0, \mathfrak{g}_1)$ and $\alpha(\mathfrak{h}_0, \mathfrak{h}_1)$ are then thought of as belonging to the same information set. Moreover, the Boolean notion of a strategy makes that the player concerned has to make the same choice with respect to a in both of them.

For an example, $a^1(b^0(\mathbf{1}, \mathbf{0}), b^0(\mathbf{0}, \mathbf{1}))$ denotes a game in which 0 is ignorant as to which choice player 1 made with respect to a^1. This particular game could be seen as representing (the information structure of) the well-known and simple game of Matching Pennies (cf. Osborne and Rubinstein [1994], p.17). In this game the two players both put a penny with either heads or tails up. If they they both turned the same side up, player 1 collects both pennies and wins. Otherwise player 0 wins with a netto profit of one penny. Compare Figure 6.1, below. The left-hand graph depicts the game in traditional fashion. The outcome $\binom{1}{-1}$ means a win for 1 and a loss for 0 and for the outcome $\binom{-1}{1}$ *vice versa.* A dotted line between two vertices indicates they belong to the same information set. The right-hand picture is the corresponding Boolean representation, in which two vertices are labelled with the same atomic strategy if they are epistemically indistinguishable for the player concerned.

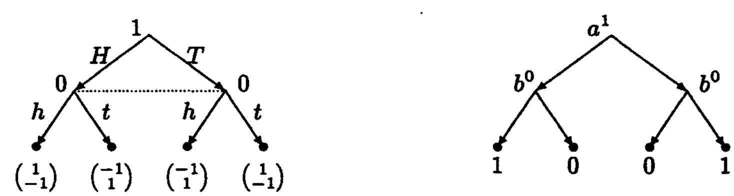

Figure 6.1

Boolean games can thus be seen to represent the information structure of a two-person zero-sum game in a remarkably neat and simple way. Games of *perfect* information can accordingly be identified with those games in which the information sets are singletons, i.e., in which each atomic strategy $\alpha \in A$ occurs at most once. A formal elaboration of the issues addressed in this section is given in Harrenstein [2001].

7 Propositional Logic

The fact that Boolean games *modulo* strategic equivalence constitute a Boolean algebra as well as the behavior of the operations $+$, $\cdot$ and $\bar{\ }$, suggests that there might be a connection with classical propositional logic. In this section, this impression is vindicated. A propositional language, $\mathcal{L}$, is introduced along with its logic. A conspicuous feature of the language is that atomic strategies figure as propositional variables. We show that the Lindenbaum algebra of the logic is isomorphic to the algebra of Boolean games, $(\mathfrak{G}_{\equiv}, +, \bullet, \bar{\ }, [\mathbf{0}], [\mathbf{1}])$.

Definition 7.1 (*Syntax of $\mathcal{L}$*)

- Propositional variables: $\alpha \in \Phi_0 = A$
- Formulae: $\varphi \in \Phi$

  $\varphi ::= \alpha \,|\, \neg\varphi \,|\, \varphi_1 \wedge \varphi_2 \,|\, \varphi_1 \vee \varphi_2$

The semantics of $\mathcal{L}$ will not surprise anyone familiar with the propositional calculus. Two matters should nevertheless be drawn some attention to. First, stripping them to their bare essentials, *valuations* for $\mathcal{L}$ are sets of propositional variables, or, alternatively, their characteristic functions with $\sigma(\alpha) = 1 \iff \alpha \in \sigma$. As the latter are identical to atomic strategies, the forcing relation, $\models$, can be construed as holding between strategy profiles and formulae. The second point is that we define the forcing relation relative to an ordering of the values 0 and 1. The idea behind this is that here these values do not so much capture a notion of truth as reflect the preferences of the two players over these values. Accordingly, we distinguish two *total* orderings over $\{0,1\}$, ω_1 and ω_0. The former is such that $0 <_{\omega_1} 1$ and mirrors the preferences of player 1, whereas in the latter $1 <_{\omega_0} 0$, which accounts for player 0's predilections. Informally, $\sigma, \omega_i \models \varphi$ betokens that φ has i's preferred value in σ. Relativizing the semantics to an order ω_i, however, affects the behavior of the binary connectives $\vee$ and $\wedge$. In the order ω_1, $\vee$ and $\wedge$ are interpreted as join and meet as usual. In ω_0 this is exactly the other way round.

Definition 7.2 (*Semantics*) Let $i \in \{0,1\}$. Define for each $\varphi \in \Phi$, and $\sigma \in \Sigma$, $\sigma, \omega_i \models \varphi$ as:

$$\begin{array}{lcl}
\sigma,\omega_i \models \alpha & :\iff & \sigma(\alpha) = i \\
\sigma,\omega_i \models \neg\varphi & :\iff & \sigma,\omega_i \not\models \varphi \\
\sigma,\omega_0 \models \varphi \wedge \psi & :\iff & \sigma,\omega_0 \models \varphi \ \text{ or } \ \sigma,\omega_0 \models \psi \\
\sigma,\omega_1 \models \varphi \wedge \psi & :\iff & \sigma,\omega_1 \models \varphi \ \ \& \ \ \sigma,\omega_1 \models \psi \\
\sigma,\omega_0 \models \varphi \vee \psi & :\iff & \sigma,\omega_0 \models \varphi \ \ \& \ \ \sigma,\omega_0 \models \psi \\
\sigma,\omega_1 \models \varphi \vee \psi & :\iff & \sigma,\omega_1 \models \varphi \ \text{ or } \ \sigma,\omega_1 \models \psi
\end{array}$$

Logical consequence, $\Gamma \models_{\omega_i} \varphi$ is defined as usual and let $\varphi \equiv_{\omega_i} \psi$ denote logical equivalence of φ and ψ. Observe that $\sigma, \omega_i \models \varphi$ iff $\sigma, \omega_{1-i} \not\models \varphi$, whence also $\varphi \models_{\omega_i} \psi$ iff $\psi \models_{\omega_{1-i}} \varphi$. Moreover, $\varphi \equiv_{\omega_i} \psi$ iff $\varphi \equiv_{\omega_{1-i}} \psi$.

Each game $\mathfrak{g} \in \mathfrak{G}$ can now be associated with a formula $\varphi \in \Phi$. Let $\perp$ be any classical contradiction, e.g., $a^0 \wedge \neg a^0$ and $\top$ any classical tautology, e.g., $a^0 \vee \neg a^0$, then define:

Definition 7.3 Define inductively for each $\mathfrak{g} \in \mathfrak{G}$ and $\alpha \in A$, $\mathsf{form}(\mathfrak{g}) \in \Phi$ as:

$$\begin{aligned}
\mathsf{form}(\mathbf{0}) &:= \bot \\
\mathsf{form}(\mathbf{1}) &:= \top \\
\mathsf{form}(\alpha(\mathfrak{g}_0, \mathfrak{g}_1)) &:= (\alpha \wedge \mathsf{form}(\mathfrak{g}_0)) \vee (\neg\alpha \wedge \mathsf{form}(\mathfrak{g}_1))
\end{aligned}$$

Now we are in a position to prove that logical notion of equivalence coincides with its game-theoretical counterpart, i.e., strategic equivalence:

Theorem 7.4 Let $i \in \{0, 1\}$. For each $\mathfrak{g} \in \mathfrak{G}$, $\sigma \in \Sigma$:

$$\begin{array}{lccc}
(i) & \sigma, \omega_i \models \mathsf{form}(\mathfrak{g}) & \iff & \mathsf{sf}(\mathfrak{g})(\sigma) = i \\
(ii) & \mathsf{form}(\mathfrak{g}) \equiv_{\omega_i} \mathsf{form}(\mathfrak{h}) & \iff & \mathfrak{g} \equiv \mathfrak{h}
\end{array}$$

Proof: The proof of (i) is by induction on the complexity of $\mathfrak{g}$. As an immediate consequence of (i) we have (ii). ⊣

To appreciate the full scope of this theorem, the reader realize that the mapping $\mathsf{form} : \mathfrak{G} \to \Phi$ is one-one. Observe the following interactions between the operations on games and the logical constants:

Proposition 7.5 Let $i \in \{0, 1\}$. For all $\mathfrak{g} \in \mathfrak{G}$:

$$\begin{aligned}
\mathsf{form}(\alpha(\mathbf{1}, \mathbf{0})) &\equiv_{\omega_i} \alpha \\
\mathsf{form}(\mathfrak{g} + \mathfrak{h}) &\equiv_{\omega_i} \mathsf{form}(\mathfrak{g}) \vee \mathsf{form}(\mathfrak{h}) \\
\mathsf{form}(\mathfrak{g} \cdot \mathfrak{h}) &\equiv_{\omega_i} \mathsf{form}(\mathfrak{g}) \wedge \mathsf{form}(\mathfrak{h}) \\
\mathsf{form}(\overline{\mathfrak{g}}) &\equiv_{\omega_i} \neg\mathsf{form}(\mathfrak{g})
\end{aligned}$$

We can now prove that the Boolean algebra of (strategic forms) of games is isomorphic to the Lindenbaum algebra $\mathfrak{L}$ associated with the logical language $\mathcal{L}$. Let $[\varphi]_{\omega_1} := \{\psi \in \Phi \mid \varphi \equiv_{\omega_1} \psi\}$ and define $\mathfrak{L} = (\Phi_{\omega_1}, \vee, \wedge, ^{-}, [\bot]_{\omega_1}, [\top]_{\omega_1})$, with $\Phi_{\omega_1} = \{[\varphi]_{\omega_1} \mid \varphi \in \Phi\}$, and $[\varphi]_{\omega_1} \vee [\psi]_{\omega_1} = [\varphi \vee \psi]_{\omega_1}$, $[\varphi]_{\omega_1} \wedge [\psi]_{\omega_1} = [\varphi \cdot \psi]_{\omega_1}$ and $\overline{[\varphi]}_{\omega_1} = [\neg\varphi]_{\omega_1}$. The proof is an easy check.

Theorem 7.6 Let $f : \mathfrak{G}_{\equiv} \to \Phi_{\omega_1}$ be such that $f([\mathfrak{g}]_{\omega_1}) = [\mathsf{form}(\mathfrak{g})]_{\omega_1}$, $f(+) = \vee$, $f(\bullet) = \wedge$ and $f(^{-}) = ^{-}$, then f is an isomorphism between $\mathfrak{G}_{\equiv}$ and $\mathfrak{L}$, i.e.,

$$f : (\mathfrak{G}_{\equiv}, +, \bullet, ^{-}, [\mathbf{0}], [\mathbf{1}]) \cong (\Phi_{\omega_1}, \vee, \wedge, ^{-}, [\bot]_{\omega_1}, [\top]_{\omega_1}).$$

8 Relativized Validity and Satisfiability

In the development of mathematical logic, also when related to the study of games, the semantical notions of validity and satisfiability play a prominent role. From the standpoint of Boolean Games, however, validity and satisfiability as such are not particularly interesting. Validity of a formula with respect to an order ω_i signifies that the corresponding game cannot otherwise but result in a victory for player i. Satisfiability in ω_i, on the other hand, conveys that there is some strategy profile such that if it is played player i wins. This, however, may require the cooperation of the opponent, which she might be reluctant to offer.

Strategic reasoning, it would seem, is rather about what a player can achieve relative to the strategies her opponent may choose. In $\mathcal{L}$, the set of propositional variables Φ_0 is divided in the set of atomic strategies of player 0, A_0, and the those of player 1, A_1. In semantic terms, what we are interested in is the possibilities of satisfying a formula given any fixed values, or rather given any values, for the atomic strategies of one of the players.

With these considerations at the back of our mind, the notions of satisfiability and validity are here generalized by parameterizing them with a subset $\Delta \subseteq \Phi_0$. This opens up a whole spectrum of notions of validity and satisfiability, of which the traditional ones are borderline cases. Moreover, these relativized semantical notions are in a natural way correlated to some strategic concepts of Boolean games.

Definition 8.1 (*Relativized Satisfiability and Validity*) Let $\sigma \sim_\Delta \sigma' :\iff \forall \alpha \in \Delta : \sigma(\alpha) = \sigma'(\alpha)$. Then define for any $\Delta \subseteq \Phi_0$ and $\varphi \in \Phi$:

φ is Δ-*independently satisfiable* in ω_i $:\iff$ $\forall \sigma \in \Sigma, \exists \sigma' \in \Sigma : \sigma \sim_\Delta \sigma' \ \& \ \sigma', \omega_i \models \varphi$

φ is Δ-*dependently valid* in ω_i $:\iff$ $\exists \sigma \in \Sigma, \forall \sigma' \in \Sigma : \sigma \sim_\Delta \sigma' \Longrightarrow \sigma', \omega_i \models \varphi$

Intuitively, Δ-independent satisfiability of a formula φ pertains to the possibility to find, for any values for the propositional variables in Δ, a valuation σ' that satisfies φ by only varying on the values of the propositional variables outside Δ. A formula φ is Δ-dependently valid if one can choose values for the proposition variables in Δ such that φ is satisfied in any valuation that respects that choice. Note that Δ-independent satisfiability implies satisfiability and that validity implies Δ-dependent validity. The converse claims, however, do not hold in general. The following fact gives an impression of how relativized satisfiability and relativized validity relate to the more traditional notions of satisfiability and validity.

Fact 8.2 For all $\varphi \in \Phi$, $i \in \{0,1\}$:

(*i*) φ is satisfiable in ω_i $\iff$ φ is $\varnothing$-independently satisfiable in ω_i

(*ii*) φ is satisfiable in ω_i $\iff$ φ is Φ_0-dependently valid in ω_i

(*ii*) φ is valid in ω_i $\iff$ φ is Φ_0-independently satisfiable in ω_i

(*iv*) φ is valid in ω_i $\iff$ φ is $\varnothing$-dependently valid in ω_i

More in general, we also have the following proposition concerning the way relativized validity and satisfiability interact. Let $\overline{\Delta} := \Phi_0 \backslash \Delta$:

Proposition 8.3 For all $\varphi \in \Phi$, $\Delta \subseteq \Phi_0$, $i \in \{0,1\}$:

(*i*) φ is Δ-dependently valid in ω_i $\Longrightarrow$ φ is $\overline{\Delta}$-independently satisfiable in ω_i

(*ii*) φ is Δ-independently satisfiable in ω_i $\iff$ φ is not Δ-dependently valid in ω_{1-i}

Proof: Consider arbitrary $\varphi \in \Phi$, $\Delta \subseteq \Phi_0$, $i \in \{0,1\}$:

(*i*) Assume φ be Δ-dependently valid in ω_i, i.e., some $\sigma \in \Sigma$ is such that for all $\sigma' \in \Sigma$, if $\sigma \sim_\Delta \sigma'$ then $\sigma', \omega_i \models \varphi$. Consider this σ as well as an arbitrary $\sigma' \in \Sigma$. Now define $\sigma^* \in \Sigma$ such that for all $\alpha \in \Phi_0$:

$$\sigma^*(\alpha) := \begin{cases} \sigma(\alpha) & \text{if } \alpha \in \Delta \\ \sigma'(\alpha) & \text{if } \alpha \in \overline{\Delta} \end{cases}$$

Clearly, $\sigma^* \sim_\Delta \sigma$ and by Δ-dependent validity of φ in ω_i, $\sigma^*, \omega_i \models \varphi$. Moreover, $\sigma' \sim_{\overline{\Delta}} \sigma^*$. Hence, σ^* is a proper witness to the $\overline{\Delta}$-independent satisfiability in ω_i of φ.

(*ii*) Consider the following equivalences:

φ is Δ-independently satisfiable in ω_i

$\iff$ for all σ, there is a σ' such that $\sigma \sim_\Delta \sigma' \ \& \ \sigma', \omega_i \models \varphi$

$\iff$ for all σ, there is a σ' such that $\sigma \sim_\Delta \sigma' \ \& \ \sigma', \omega_{1-i} \not\models \varphi$

$\iff$ there is no σ such that for all σ' : $\sigma \sim_\Delta \sigma' \Longrightarrow \sigma', \omega_{1-i} \models \varphi$

$\iff$ φ is not Δ-dependently valid in ω_{1-i} $\dashv$

We argued that the traditional logical concepts validity and satisfiability do not chime in particularly well with interesting strategic notions for Boolean games. Relativized validity and satisfiability fare considerably better in this respect. Relativized validity has for instance a natural game-theoretical counterpart in a player having a winning strategy in a Boolean game.

Fact 8.4 For all $\mathfrak{g} \in \mathfrak{G}$, $i \in \{0,1\}$:
i has a winning strategy in $\mathfrak{g}$ $\iff$ $\mathsf{form}(\mathfrak{g})$ is A_i-dependently valid in ω_i

Proof: Consider an arbitrary $\mathfrak{g} \in \mathfrak{G}$ and let $i \in \{0,1\}$.

$\Rightarrow$: Assume player i have a winning strategy in $\mathfrak{g}$. Then some $s \in S_i$ is such that for all $s' \in S_{1-i}$: $\mathsf{sf}(\mathfrak{g})(s \cup s') = i$. Consider this s and let $\sigma = s$. Also consider an arbitrary $\sigma' \in \Sigma$ such that $\sigma \sim_{A_i} \sigma'$. Obviously, there is an $s'' \in S_{1-i}$ such that $\sigma' = s \cup s''$. Since s is winning, we have $\mathsf{sf}(\mathfrak{g})(\sigma') = i$. Whence, by Theorem 7.4, $\sigma', \omega_i \models \mathsf{form}(\mathfrak{g})$.

$\Leftarrow$: Assume $\mathsf{form}(\mathfrak{g})$ be A_i-dependently valid in ω_i. By definition there is a $\sigma \in \Sigma$ such that for all $\sigma' \in \Sigma$, if $\sigma \sim_{A_i} \sigma'$ then $\sigma', \omega_i \models \varphi$. Consider this σ and let $s = \{\alpha \in A_i \mid \alpha \in \sigma\} \in S_i$. For arbitrary $s' \in S_{1-i}$, obviously, $\sigma \sim_{A_i} s \cup s'$. Hence $s \cup s', \omega_i \models \mathsf{form}(\mathfrak{g})$. Applying theorem 7.4 ($i$) we finally have $\mathsf{sf}(\mathfrak{g})(s \cup s') = i$. ⊣

To what game-theoretical concept relativized satisfiability corresponds might be slightly more elusive. In any case, $\mathsf{form}(\mathfrak{g})$ is A_{1-i}-independently satisfiable in ω_i if whenever i knows her opponent's strategy she can choose her strategy to win the game. Of course, this holds whenever i has a winning strategy. This observation is a straight instance of Proposition 8.3(i), above.

The converse of this proposition, it be noted, does not hold in general, but it does entertain an intimate connection with the game-theoretical notion of determinacy. A game is said to be *determined* if one of the players has a winning strategy. As matters turn out, the games for which the converse of Proposition 8.3 holds are exactly the games that are determined. Hence we obtain the following theorem:

Theorem 8.5 For all $\mathfrak{g} \in \mathfrak{G}$, $i \in \{0,1\}$:
$\mathfrak{g}$ is determined if and only if the following two claims are equivalent:

(1) $\mathsf{form}(\mathfrak{g})$ is A_i-dependently valid in ω_i

(2) $\mathsf{form}(\mathfrak{g})$ is A_{1-i}-independently satisfiable in ω_i

Proof: Consider an arbitrary game $\mathfrak{g} \in \mathfrak{G}$ as well as an arbitrary $i \in \{0,1\}$.

$\Rightarrow$: Assume $\mathfrak{g}$ to be determined. The (1) $\Rightarrow$ (2) direction follows immediately from Proposition 8.3(i). For the (2) $\Rightarrow$ (1) direction, assume $\mathsf{form}(\mathfrak{g})$ to be A_{1-i}-independently satisfiable in ω_i. With determinacy of $\mathfrak{g}$ and Fact 8.4, either (a) $\mathsf{form}(\mathfrak{g})$ is A_i-dependently valid in ω_i, or (b) $\mathsf{form}(\mathfrak{g})$ is A_{1-i}-dependently valid in ω_{1-i}. By the assumption and Proposition 8.3(ii), the latter cannot be. So, as not (b), (a): $\mathsf{form}(\mathfrak{g})$ is A_i-dependently valid in ω_i.

$\Leftarrow$: Assume (1) and (2) to be equivalent. First assume that $\mathsf{form}(\mathfrak{g})$ is A_i-dependently valid in ω_i. Then, with Fact 8.4, i has a winning strategy in $\mathfrak{g}$. Now assume that $\mathsf{form}(\mathfrak{g})$ is not A_i-dependently valid in ω_i. In this case, by assumption, $\mathsf{form}(\mathfrak{g})$ is not A_{1-i}-independently satisfiable in ω_i either. Applying Proposition 8.3(ii), $\mathsf{form}(\mathfrak{g})$ is A_{1-i}-dependently valid in ω_{1-i}, and so $1-i$ has a winning strategy in $\mathfrak{g}$. ⊣

As early as 1913 Zermelo (Zermelo [1913]) proved finite two-person strictly competitive games of perfect information to be determined. This result naturally propagates to Boolean games of perfect information (cf. Section 6). Some games of imperfect information are determined as well but this fact does not hold in general. Theorem 8.5 also applies to these games and as such features a logical characterization of determinacy.

9 Conclusion

In this paper we introduced a framework of finite, two-person and fully competitive games, including a calculus enabling one to find winning strategies for the players. Within this framework both games of perfect information and games of imperfect information can elegantly be represented. Operations on games were defined and we demonstrated that *modulo* a suitable notion of strategic equivalence these games constitute a Boolean algebra. We proved this algebra to be isomorphic to the Lindenbaum algebra of an appropriate propositional logic, thus facilitating a *logical* approach to game-theoretical issues. Finally, we generalized the concepts of validity and satisfiability and to these notions precise game-theoretical readings were attached. This enabled us to characterize determined games in logical terms.

In the future we aim to extend the framework to multi-player games and to develop a logic corresponding to it. We expect this also to have interesting repercussions for the interpretations of established logical notions.

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