Ambiguity Made Precise: A Comparative Foundation (Extended Abstract)[1]

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There is no doubt that the subjective expected utility (SEU) theory of decision making under uncertainty of Savage (1954) is solidly established as the choice-theoretic underpinning of modern economic theory. However, Ellsberg (1961)'s famous thought experiment shows that SEU cannot take into account the possibility that the information a decision maker (DM) has about some uncertain event (that is relevant to her choice) be vague or imprecise, and that such "ambiguity" might affect her behavior.

One of the most successful extensions of SEU is the Choquet expected utility (CEU) model introduced by Schmeidler (1989). Because greater generality entails less structure, the CEU model is nowhere as theoretically malleable as SEU, but it has been studied by economists interested in understanding phenomena at odds with SEU. However, as Epstein (1997) observes, the CEU model still lacks a general theoretical foundation for the notion of ambiguity aversion, akin to that developed for risk aversion in the SEU case. The objective of this paper is to provide such a foundation: We present a behavioral definition of ambiguity aversion for Savage's set-up, and we characterize it formally.

To understand how we approach the problem, it is useful to go back to the characterization of risk aversion in the SEU model. The following general approach to defining risk aversion was outlined by Yaari (1969). For a given state space S, imagine a collection of acts $\mathcal{F}$ which map S into $\mathbb{R}$ (e.g., into monetary payoffs). Start by defining a *comparative* notion of risk aversion: Say that the SEU preference $\succcurlyeq_2$ is more risk averse than $\succcurlyeq_1$ if both induce the same beliefs μ on S, and the following two implications hold for every constant act x and every uncertain act f:

$$x \succcurlyeq_1 f \quad \Rightarrow \quad x \succcurlyeq_2 f \tag{1}$$

$$x \succ_1 f \quad \Rightarrow \quad x \succ_2 f \tag{2}$$

(where $\succ$ is the asymmetric component of $\succcurlyeq$). Observe that, to be sure that only differences in risk attitude are observed, we have to compare DMs with identical beliefs. If that were not the case, we might confuse different risk attitudes with different beliefs. Once the comparative relation is defined, all that remains is to decide who is a *risk neutral* DM. For instance, it can be an expected value maximizer. We then define *risk averse* a DM who is more risk averse than a risk neutral one. Yaari (1969) showed that this leads to the usual concavity characterization.

Note that this construction is based on two arbitrary choices. First, we established constant acts as "riskless." Second, we established expected value maximization as the benchmark for defining risk aversion. (While both these choices are quite natural in our setting, they might not be more generally. For instance, what is the natural benchmark when prizes are two-dimensional?) But an important feature of this approach is that it is *behavioral.* That is, to decide which of two DMs is more risk averse we only need to observe their preferences. No explicit knowledge of the DM's utility function or beliefs is required.

In this work, we apply a similar procedure to describe ambiguity attitude: We start from a comparative notion of "more ambiguity averse than...," and then establish a benchmark, thus obtaining an "absolute" definition of ambiguity aversion.

Our development of the "more ambiguity averse..." relation starts from the following intuitive considerations:

1. If a DM prefers an unambiguous act to an ambiguous one, a more ambiguity averse one will do the same.

2. If a DM prefers an ambiguous act to an unambiguous one, a less ambiguity averse one will do the same.

While this seems very natural, the important question is of course: What do we mean by "ambiguous/unambiguous" acts? A tempting idea is to use the *weakest* preconceived notion of "unambiguous" act: Say that an act is unambiguous if it is a *constant*, and ambiguous otherwise. This is tantamount to saying that $\succcurlyeq_2$ is more ambiguity averse than $\succcurlyeq_1$ when Eqs. (1) and (2) hold. But, as in defining risk aversion it is necessary to factor out differences in beliefs (cf. Yaari (1969, p.317)), here we have to be careful to rule out differences in risk attitude that might intrude in the comparison.

To avoid this problem, we develop a behavioral condition that *isolates ambiguity attitudes from risk attitudes*, by insuring that the DMs' risk attitudes are equal (i.e., they have the same von Neumann-Morgenstern utility function, up to affine transformations). This condition has two nice features. First, it is very general: It only requires that the set of consequences be rich enough to identify precisely the utility function (the state space can instead be arbitrary). Second, it imposes *no restrictions* on the ambiguity attitudes of the DMs' (since it is built only on comparisons among comonotonic acts).

Our definition of comparative ambiguity aversion is therefore the conjuction of (1) and (2) with the behavioral condition that insures identity of utilities. Notice that this identity does *not* limit the scope of the absolute definition of ambiguity aversion. In fact, the latter is conceptually based on a comparison of the DM with a replica of herself, differing only in her "willingness to bet" on relevant events.

The second step is the choice of a benchmark against which to measure ambiguity aversion. We opt for what seems to us the natural candidate: SEU preferences. We thus call *ambiguity averse* a preference relation $\succcurlyeq$, for which there is a SEU preference (with the same risk attitude) which is less ambiguity averse than $\succcurlyeq$. (Ambiguity love and neutrality are then defined in the obvious way.)

With the definition of ambiguity aversion in our hands, we move on to the central results of the paper, characterizing formally the class of capacities which represent ambiguity averse preferences. We show that such characterization is very clean: A preference is ambiguity averse if and only if it is represented by a capacity with a nonempty core (also called a *balanced* capacity), i.e., it is dominated by a probability. The only condition used in proving this result (other than the CEU representation) is the richness of consequence space mentioned above.

The characterization of ambiguity love and neutrality follow immediately in a smilar fashion (it is nice to know that the only ambiguity neutral preferences in this sense are the SEU ones). We also provide an interesting characterization of the comparative ambiguity relation. The fact that balancedness characterizes ambiguity aversion might seem surprising at first. In fact, convexity of the capacity seems the natural analogue of the concavity of utility that characterizes risk aversion (also because this is the case for Schmeidler (1989)'s definition). We explain that the reason for this failure of intuition is the difference in the uniqueness properties of utilities and capacities.

The final sections of the paper apply the definition to Ellsberg's classic 3-color example, and then compare the approach taken in this paper to Schmeidler (1989)'s, and to the recent Epstein (1997), which starts from similar premises and gets to very different conclusions.

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