# Congestion Games with Variable Demands

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## ABSTRACT

We initiate the study of *congestion games with variable demands* where the (variable) demand has to be assigned to exactly one subset of resources. The players' incentives to use higher demands are stimulated by non-decreasing and concave utility functions. The payoff for a player is defined as the difference between the utility of the demand and the associated cost on the used resources. Although this class of non-cooperative games captures many elements of real-world applications, it has not been studied in this generality, to our knowledge, in the past.

We study the fundamental problem of the existence of *pure Nash equilibria* (PNE for short) in congestion games with variable demands. We call a set of cost functions $\mathcal{C}$ *consistent* if every congestion game with variable demands and cost functions in $\mathcal{C}$ possesses a PNE. We say that $\mathcal{C}$ is *FIP consistent* if every such game possesses the $\alpha$-Finite Improvement Property for every $\alpha > 0$. Our main results provide a complete characterization of consistency of cost functions revealing structural differences to congestion games with *fixed* demands (weighted congestion games), where in the latter even inhomogeneously exponential functions are FIP consistent.

Finally, we study consistency and FIP consistency of cost functions in a slightly different class of games, where every player experiences the same cost on a resource (uniform cost model). We give a characterization of consistency and FIP consistency showing that only homogeneously exponential functions are consistent while no functions are FIP consistent.

## Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics

## General Terms

Theory

## Keywords

Congestion Games, Variable Demand, Pure Nash Equilibria

## 1. INTRODUCTION

Resource allocation problems play a key role in many applications. Whenever a set of demands needs to be satisfied by scarce resources, the goal is to find the most profitable or least costly allocation of the resources to the demands. Examples are traffic networks Beckmann et al. (1956); Roughgarden (2005); Smith (1979); Wardrop (1952) and telecommunication networks Johari and Tsitsiklis (2006); Kelly et al. (1998); Srikant (2003). A common characteristic of these examples is that the allocation of resources is determined by a finite number of independent players, each optimizing an individual objective function. A natural framework for analyzing such non-cooperative games are *congestion games* as introduced by Rosenthal (1973). In a congestion game, there is a set of resources and a pure strategy of a player consists of a subset of resources. The cost of a resource depends only on the number of players choosing the resource, and the private cost of a player is the sum of the costs of the chosen resources. Under these assumptions, Rosenthal proved the existence of a pure Nash equilibrium (PNE for short).

In the past, the existence of PNE has been analyzed in many variants of congestion games such as scheduling, routing, facility location, network design, each variant with unweighted and weighted players, see Ackermann et al. (2009); Anshelevich et al. (2008); Chen and Roughgarden (2009); Gairing et al. (2006); Ieong et al. (2005); Milchtaich (2006). Most of these previous works have a common feature: given a set of resources whose cost increases with increasing congestion, every player allocates a *fixed* demand to an available subset. While obviously important, such models do not take into account a fundamental property of many real-world applications: the intrinsic coupling between the quality or cost of the resources and the resulting demands for the resources. A prominent example of this coupling is the flow control problem in telecommunication networks. In this setting, players receive a non-negative utility from sending data and the perceived costs increase with congestion. In this and other examples, the demands will be reduced if the resources are congested, and increased if the resources are uncongested. Allowing for *variable demand* is, thus, a natural prerequisite for modeling the tradeoff between the benefit for demand, and the quality of the resource.

There is a large body of work addressing the issue of variable demands: Cole et al. (2006); Low and Lapsley (1999); Kelly et al. (1998); Shenker (1995); Srikant (2003) in the

context of telecommunication networks and Beckmann et al. (1956); Haurie and Marcotte (1985) in the context of traffic networks. Most of these works assume that the (variable) demand may be *fractionally* distributed over the available subsets of resources. This assumption together with convexity assumptions on the cost and utility functions implies the existence of a PNE by Rosen's theorem (Rosen (1965)). Allowing a fractional distribution of the demand, however, is obviously not possible in many applications. For instance, the standard TCP/IP protocol suite uses single path routing, because splitting the demand comes with several practical complications, e.g., packets arriving out of order, packet jitter due to different paths delays etc.

We initiate the study of congestion games with variable demands, where the (variable) demand has to be assigned to exactly one subset of resources. We impose standard economic assumptions (cf. Haurie and Marcotte (1985); Kelly et al. (1998); Shenker (1995)) that every player is associated with a non-decreasing and concave utility function measuring the utility for the demand. The payoff for a player is defined as the difference between the utility and the associated congestion cost on the used resources where the cost of a player on a resource is the product of the cost function, and the player's demand.

There are two fundamental goals from a system design perspective: ($i$) the system must be *stabilizable*, that is, there must be a stable point (PNE) from which no player wants to unilaterally deviate; ($ii$) myopic play of the players should guide the system to a *stable* state. Because the utility functions are only known to the players (as they are private information) and not available to the system designer, we study the above two issues with respect to the used *cost functions* (which represent the technology associated with the resources, e.g., queuing discipline at routers, latency function in transportation networks, etc.).

Let $\mathcal{C}$ be a set of cost functions and let $\mathcal{G}(\mathcal{C})$ be the set of *all* congestion games with variable demands and cost functions in $\mathcal{C}$. We say that $\mathcal{C}$ is *consistent* if every game in $\mathcal{G}(\mathcal{C})$ possesses a PNE. We say that $\mathcal{C}$ is *FIP consistent* if for every game in $\mathcal{G}(\mathcal{C})$, every sequence of unilateral $\alpha$-improvements (improvements that add at least $\alpha$ to the payoff of the deviating player, see Monderer and Shapley (1996)) is finite. The main goal of this work is to investigate consistency and the $\alpha$-FIP of the used cost functions.

### Connection to Prior Work.

Already earlier work investigated the impact of the *cost structure* on the existence of PNE and the $\alpha$-FIP in variants of congestion games, see Altman et al. (2002); Fotakis et al. (2005); Libman and Orda (2001); Orda et al. (1993). In a previous paper (Harks and Klimm, 2010), we characterized the set of cost functions that are consistent w.r.t. weighted congestion games. Specifically, for this class of games a set of cost functions is consistent if and only if it either contains only affine functions or certain exponential functions, see Table 1.

The existence of PNE with respect to the cost structure also plays a fundamental role in the vast literature on Cournot oligopoly games in the economic theory of imperfect competition. In a Cournot oligopoly game, there is a set of firms each producing quantities so as to satisfy an elastic demand. The production cost for every player is modeled by a convex cost function and the interaction between the

firms comes from the price determination mechanism which is dependent on the total supply on the market. Note that congestion games with variable demands can be interpreted as a natural generalization of the Cournot oligopoly game. In Johari and Tsitsiklis (2005) it is proved that Cournot oligopoly games are basically equivalent (in terms of the set of Nash equilibria) to congestion games with variable demand and a single resource (which are termed Cournot oligopsonies in Johari and Tsitsiklis (2005)). The model proposed in this article is more general since the strategy space of players involves not only quantities but also sets of allowable subsets of resources (markets).

The pioneering work of Cournot (1838) established the existence of an equilibrium point for this model assuming zero costs and concave inverse demand functions. Further existence results for more general cost functions have been established by many researchers, among others Novshek (1985); Roberts and Sonnenschein (1976).

In the seminal paper by Orda et al. (1993), the authors address the issue of uniqueness of PNE in congestion games with fixed splittable demands. They give sufficient conditions for uniqueness of PNE for several classes of cost functions. In the final section of their paper, the following question is raised (we quote from the paper):

> "Several other open questions of practical value deserve attention. For example, in many networks users are restricted to route their flow along a single path with strict rules of changing them. Under such circumstances an NEP may not exist at all and complicated oscillatory behavior is likely to arise."

### Our Results.

We initiate the study of congestion games with variable demands and focus on the existence of pure Nash equilibria and the finite improvement property. Our main results are structural characterizations of the existence of PNE and the $\alpha$-FIP with respect to the cost structure. Specifically, we show the following:

Let $\mathcal{C}$ be a set of non-negative, strictly increasing and twice differentiable cost functions. We prove that $\mathcal{C}$ is consistent if and only if exactly one of the following cases hold: (a) $\mathcal{C}$ contains only affine functions $c(\ell) = a_c\,\ell + b_c$ with $a_c > 0$, $b_c \geq 0$; (b) $\mathcal{C}$ contains only homogeneously exponential functions such that $c(\ell) = a_c\,e^{\phi\ell}$ for some $a_c, \phi > 0$, where $a_c$ may depend on $c$, while $\phi$ must be equal for all $c \in \mathcal{C}$. Moreover, we characterize the $\alpha$-FIP in congestion games with variable demands. We prove that $\mathcal{C}$ is FIP consistent if and only if $\mathcal{C}$ contains only affine functions. Note that this implies that congestion games with elastic demands and non-affine cost do not possess a (generalized ordinal) potential function, in general. The formal results appear as Theorems 2 and 3. We show that our results remain valid for games with network structure and, thus, our characterizations settle the questions raised by Orda et al. (1993) for network congestion games involving variable demands.

We then investigate a slightly different class of games known as *uniform* games. They differ from the previously studied games in the definition of the players' payoff functions. In uniform games the cost for a player on a resource is *not* multiplied with the demand of that player. Uniform games haven been first introduced by Koutsoupias and Pa-

**Table 1: Existence of PNE and the $\alpha$-FIP in congestion games with variable demands with proportional costs and uniform costs. Note the fundamental structural difference to weighted congestion games (with fixed demands) where in the proportional and uniform cost model there is always a PNE and the $\alpha$-FIP for affine and inhomogeneously exponential cost functions. The results (1) are from Fotakis et al. (2005), the results (2) are from Panagopoulou and Spirakis (2006), results (3) and (4) are from Harks et al. (2009) and Harks and Klimm (2010), respectively. All other results of this table are presented in this paper.**

| costs | var. demands prop. costs PNE | $\alpha$-FIP | var. demands unif. costs PNE | $\alpha$-FIP | fix. demands prop. & unif. PNE | $\alpha$-FIP |
|---|---|---|---|---|---|---|
| affine | yes | yes | no | no | yes (1) | yes (1) |
| hom. exp. | yes | no | yes | no | yes (2) | yes (2) |
| inhom. exp. | no | no | no | no | yes (3) | yes (3) |
| other | no | no | no | no | no (4) | no (4) |

padimitriou (1999) in the context of scheduling games (also known as the KP-model), see also several follow-up papers, e.g., Fotakis et al. (2005); Gairing et al. (2006); Ieong et al. (2005); Milchtaich (2006). As noted by Koutsoupias and Papadimitriou (1999), uniform costs are motivated by scheduling applications, where the cost function is frequently used to model the achieved makespan which is (under round-robin processing) equal for every job on the same resource. Uniform cost structures play also a key role in telecommunication networks. In large-scale telecommunication networks, it is highly desirable to charge every player the *same* cost regardless of the actual resource consumption of every player, because every resource needs only to communicate a single value to the players giving rise to an efficient and scalable implementation, see Johari and Tsitsiklis (2009) and Srikant (2003).

Our second main result provides a complete characterization of consistency of a set of cost functions for the uniform cost model. We prove that $\mathcal{C}$ is consistent w.r.t. uniform cost games if and only if $\mathcal{C}$ contains homogeneously exponential functions such that $c(\ell) = a_c\, e^{\phi\ell}$ for some $a_c, \phi > 0$, where $a_c$ may depend on $c$, while $\phi$ must be equal for all $c \in \mathcal{C}$. Surprisingly, this characterization reveals that uniform games need not possess a PNE, even if costs are affine. We also characterize the $\alpha$-FIP in the uniform cost model. We prove that $\mathcal{C}$ is FIP consistent if and only if $\mathcal{C} = \emptyset$. For the case of homogeneously exponential cost functions, however, we derive an improvement dynamic converging to a PNE, thus, showing that the improvement graphs of the resulting games are weakly acyclic.

Our results are summarized in Table 1.

*Main Ideas and Outline.*

After introducing the basic model in Section 2, we prove the "only if" direction of our first result (Theorem 2) in Section 3. In the proof, we first establish a connection between weighted congestion games and congestion game with vari-

able demands. Given a weighted congestion without PNE, we derive a congestion game with variable demands using the same cost functions that also has no PNE. The proof idea relies on a careful design of feasible (concave and differentiable) utility functions which preserve the improvement cycles of the original weighted congestion game. Thus, we can use an earlier result of Harks and Klimm (2010) stating that a set of cost functions is consistent w.r.t. weighted congestion games if and only if this set contains either affine functions or certain exponential functions. The hard part of completing the "only if" direction lies in excluding inhomogeneously exponential cost functions. We prove that these functions are not consistent by studying a class of congestion games with fixed *resource dependent demands*. We identify a subclass of these games with inhomogeneously exponential cost functions for which we construct a congestion game with variable demands. We show that the thus constructed game fulfills the invariant of preserved improvement cycles with respect to the original game. While this part of the proof is perhaps the most technical, we obtain as a side-product of our analysis a novel characterization of the existence of PNE for congestion games with resource dependent demands showing that only affine functions are consistent.

In Section 4, the "if" part is proven. We introduce a novel potential function concept that we term *essential generalized ordinal potentials*. The idea is to require that there is a real-valued function that must increase only for a subset of improving moves. We further introduce *local* essential potentials, where this property must hold only for a global maximum of the potential. For games with homogeneously exponential cost functions, we derive a *local essential potential* completing the "if" direction (games with affine costs are exact potential games). In Section 5, we investigate a class of games that we term *uniform congestion games with variable demands*. We give similar characterizations of consistency of cost functions, yielding that only homogeneously exponential functions are consistent. We prove the "if" direction by deriving an essential potential. We conclude the paper in Section 6 by presenting new research directions.

## 2. PRELIMINARIES

Congestion games with variable demands are strategic games $G = (N, \bar{X}, (\pi_i)_{i\in N})$, where $N = \{1, \ldots, n\}$ is the non-empty and finite set of players, $\bar{X} = \times_{i\in N} \bar{X}_i$ is the non-empty set of *strategy profiles*, and $\pi_i : \bar{X} \to \mathbb{N}$ is the *individual payoff* function that specifies the payoff value of player $i$ for each state $\bar{x} \in \bar{X}$. We define strategies and payoff functions using the general notion of a congestion model. A tuple $\mathcal{M} = (N, R, X, (c_r)_{r\in R})$ is called a *congestion model* if $N$ is a set of players, $R$ is a finite set of *resources*, and $X = \times_{i\in N} X_i$ where $X_i \subseteq 2^R$ is the set of *configurations* available to player $i$. Every resource $r \in R$ is endowed with a cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. In a congestion game with variable demands, every player $i \in N$ is allowed to choose a configuration $x_i \in X_i$ and a non-negative demand $d_i \in \mathbb{R}_{\geq 0}$ that she places on all resources in $x_i$. The incentive to use higher demands is stimulated by a utility function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that measures the benefit that player $i$ receives from choosing a certain demand. Note that this benefit is independent of the resources chosen and depends solely on the chosen demand.

Let $\mathcal{M} = (N, R, X, (c_r)_{r\in R})$ be a congestion model and let $(U_i)_{i\in N}$ be a collection of utility functions. We define

a congestion game with variable demands and *proportional costs* as the game $G(\mathcal{M}) = (N, \bar{X}, \pi)$, where $\bar{X} = (X, \mathbb{R}_{\geq 0})$, $\pi = (\pi_i)_{i \in N}$ and $\pi_i(x, d) = U_i(d_i) - \sum_{r \in x_i} d_i c_r(\ell_r(x, d))$, and $\ell_r((x, d)) = \sum_{j \in N : r \in x_j} d_j$. We call $\ell_r(x, d)$ the *load* on resource $r$ under strategy $(x, d)$. Note that these games are maximization games. A configuration $x \in X$ together with a demand profile $d \in \mathbb{R}_{\geq 0}^n$ forms a strategy profile $\bar{x} = (x, d)$. Note that proportional costs have been used before in Goemans et al. (2005) for the case of weighted congestion games.

We use standard game theory notation; for a set $S \subseteq N$ we denote by $-S$ its complement and by $\bar{X}_S = \times_{i \in S} \bar{X}_i$ we denote the set of strategy profiles of players in $S$. Instead of $\bar{X}_{-\{i\}}$ we will write $\bar{X}_{-i}$, and with a slight abuse of notation we will sometimes write a strategy profile as $\bar{x} = (\bar{x}_i, \bar{x}_{-i})$ meaning that $\bar{x}_i \in \bar{X}_i$ and $\bar{x}_{-i} \in \bar{X}_{-i}$.

For a constant $\alpha \geq 0$, a pair $(\bar{x}, (\bar{y}_i, \bar{x}_{-i})) \in \bar{X} \times \bar{X}$ is called an $\alpha$-*improving move* of player $i$ if $\pi_i(\bar{x}) + \alpha < \pi_i(\bar{y}_i, \bar{x}_{-i})$. We denote by $I^\alpha(i)$ the set of $\alpha$-improving moves of player $i \in N$, and we set $I^\alpha = \bigcup_{i \in N} I^\alpha(i)$. We call a sequence of strategy profiles $\gamma = (x^0, x^1, \dots)$ an $\alpha$-*improvement path* if every tuple $(x^k, x^{k+1}) \in I^\alpha$. A strategy profile $\bar{x} \in \bar{X}$ is a *pure Nash equilibrium*, or PNE for short, if $(\bar{x}, \bar{y}) \notin I^0$ for all $\bar{y} \in \bar{X}$. $G$ has the $\alpha$-*finite improvement property* ($\alpha$-FIP) if every $\alpha$-improvement path is finite. Let $\mathcal{C}$ be a class of cost functions. We call $\mathcal{C}$ *consistent w.r.t. congestion games with variable demands* (or simply *consistent*) if every congestion game with variable demands and cost functions in $\mathcal{C}$ admits a PNE. $\mathcal{C}$ is *FIP consistent* if every congestion game with variable demands and cost functions in $\mathcal{C}$ has the $\alpha$-FIP.

The following two assumptions contain mild restrictions on feasible utility functions and cost functions and are standard in the literature, see Haurie and Marcotte (1985); Kelly et al. (1998); Shenker (1995).

*Assumption 1.* For every resource $r \in R$ the cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is twice continuously differentiable and strictly increasing.

We denote by $\mathcal{C}^2(\mathbb{R}_{\geq 0})$ the set of functions satisfying Assumption 1.

*Assumption 2.* For every player $i \in N$ the utility function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is differentiable, non-decreasing and concave.

*Remark 1.* In contrast to most of the works in the area of Cournot games or congestion games with splittable demands (e.g., Haurie and Marcotte (1985); Kelly et al. (1998); Orda et al. (1993)), we do not assume semi-convexity of cost functions.

## 3. NECESSARY CONDITIONS

At first we present some simple but useful observations. It is easy to see that consistent cost functions cannot have bounded marginal costs. Formally, if $\mathcal{C}$ is consistent, then $c(x) + xc'(x) \to \infty$ as $x \to \infty$ for all $c \in \mathcal{C}$. To see this, assume that there is $c \in \mathcal{C}$ with $c(x) + xc'(x) < M$ for some $M \in \mathbb{R}_{>0}$. Consider a game with one resource and cost function $c$ and one player with utility function $U(x) = (M + 1)x$. The payoff of the player increases with the demand, thus, this game does not admit a PNE.

Moreover, the following lemma will be useful throughout this paper. It simply uses the first-order optimality condition of a PNE.

LEMMA 1. *Let $(x^*, d^*)$ be a PNE. Then, we have $U_i'(d_i^*) = \left( \partial d_i^* \sum_{r \in x_i} c_r(\ell_r(x^*, d^*)) \right) / \partial d_i^*$ for all $i \in N$ with $d_i^* > 0$, and $U_i'(d_i^*) \leq \left( \partial d_i^* \sum_{r \in x_i} c_r(\ell_r(x^*, d^*)) \right) / \partial d_i^*$ for all $i \in N$ with $d_i^* = 0$.*

### Necessity of Affine or Exponential Costs.

First, we establish a connection between congestion games with variable demands and weighted congestion games. We will exploit this connection to show that consistent cost functions for congestion games with variable demands must be either affine or exponential.

Let $\mathcal{M} = (N, R, X, (c_r)_{r \in R})$ be a congestion model and $(d_i^{\mathrm{w}})_{i \in N}$ be a vector of demands with $d_i^{\mathrm{w}} \in \mathbb{R}_{>0}$. The corresponding *weighted congestion game* is the strategic game $G^{\mathrm{w}}(\mathcal{M}) = (N, X, \pi)$, where $\pi$ is defined as $\pi = \times_{i \in N} \pi_i$, $\pi_i(x) = \sum_{r \in x_i} d_i^{\mathrm{w}} c_r(\ell_r(x))$ and $\ell_r(x) = \sum_{j \in N : r \in x_j} d_j^{\mathrm{w}}$. Note that these games are cost minimization games. We will show that whenever there is a weighted congestion game $G^{\mathrm{w}}(\mathcal{M})$ that does not admit a PNE, then there is also a congestion game with variable demands $G(\mathcal{M})$ under the same congestion model without a PNE. The proof of this result is constructive, i.e., given the weighted congestion game $G^{\mathrm{w}}(\mathcal{M})$ and the corresponding vector of demands $d_i^{\mathrm{w}}$, we construct a congestion game with variable demands not possessing a PNE.

Note that the main difficulty lies in the fact that in congestion games with variable demands the strategy space is strictly larger than in weighted congestion games. We overcome this issue by designing for every player $i$ a utility function $U_i$ that allows us to restrict a priori the set of equilibrium demands of player $i$ to a small environment $(t_i - \sigma, t_i + \sigma)$ around a *target demand* $t_i$. To this end, recall that Lemma 1 establishes that in every PNE player $i$'s marginal utility equals her marginal cost given that her demand is strictly positive. Our basic idea is to define the utility function $U_i$ of player $i$ such that the following two properties are guaranteed: $(i)$ player $i$ always has an interest to play a positive demand; $(ii)$ for every possible value $C'$ of player $i$'s marginal costs, the equation $U_i'(d_i) = C'$ is met only for $d_i \in (t_i - \sigma, t_i + \sigma)$. As we will see, we can find smoothed 2-wise linear utility functions that suit our purposes.

We define the piecewise linear function through the sequence of points $S = ((\tau_0, v_0), (\tau_1, v_1), \dots, (\tau_k, v_k))$ as the function

$$g_S(x) = \begin{cases} v_i + \frac{v_i - v_{i-1}}{\tau_i - \tau_{i-1}}(x - \tau_i), & \text{if } x \in [\tau_i, \tau_{i+1}), \\ v_k, & \text{if } x \in [\tau_k, +\infty). \end{cases} \quad (1)$$

Since we are interested in obtaining non-decreasing and concave functions we will assume in the following that the sequence of slopes $\left( \frac{v_i - v_{i-1}}{\tau_i - \tau_{i-1}} \right)_{i=1,\dots,k}$ is positive and non-increasing. To obtain differentiable functions, we choose a sufficiently small $\sigma > 0$ and replace the function $g$ on every set $(\tau_i - \sigma, \tau_i + \sigma)$, $i = 1, \dots, k$ by a non-decreasing and differentiable function such that the functional values and first derivatives in $\tau_i - \sigma$ and $\tau_i + \sigma$ comply.[1] We denote the corresponding smoothed piecewise linear function by $g^\sigma(S)$. By designing special smoothed 2-wise linear functions, we obtain the following lemma.

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[1] For instance, we can choose a suitable quadratic function on every interval $(\tau_i - \sigma, \tau_i + \sigma)$.

LEMMA 2. *Let $\mathcal{M}$ be a congestion model, $(t_i)_{i \in N} \in \mathbb{R}_{>0}^n$ a vector of target demands with $t_{\min} = \min_{i \in N}\{t_i\}$ and $t_{\max} = \max_{i \in N}\{t_i\}$. Let $M > \max\{\frac{2\sum_{r \in R} c_r(T)}{t_{\min}}, \sum_{r \in R}(c_r(T) + (t_{\max} + \sigma) \max_{\ell \in [t_{\min}, T]} c_r'(\ell))\}$, where $T = \sum_{j \in N}(t_j + \sigma)$ and $\sigma < t_{\min}/2$. Let $G$ be the congestion game with variable demands and utilities $U_i = g^\sigma((0,0), (t_i, M t_i))$ for all $i \in N$. Then, $d_i^* \in (t_i - \sigma, t_i + \sigma)$ for all $i \in N$ and every PNE $(x^*, d^*)$ of $G$.*

PROOF. For contradiction, suppose there is a PNE $\bar{x}^* = (x^*, d^*)$ of $G$ and $i \in N$ with $d_i^* \notin (t_i - \sigma, t_i + \sigma)$. First, assume $d_i^* > t_i + \sigma$. As $c_r$ is strictly increasing, we may assume without loss of generality that there is $r \in x_i$ with $c_r(d_i^*) > 0$. Lemma 1 implies $0 = U_i'(d_i^*) = \sum_{r \in x_i^*}(c_r(\ell_r(\bar{x}^*)) + d_i^* c_r'(\ell_r(\bar{x}^*)) > 0$, which is a contradiction.

Now, suppose $d_i^* = 0$ and thus $\pi_i(x^*, d^*) = 0$. We denote by $\bar{y} = (x_i^*, t_i - \sigma, x_{-i}^*, d_{-i}^*)$ the strategy profile in which player $i$ chooses her demand equal to $t_i - \sigma$ instead. Note that $\pi_i(\bar{y}) = M(t_i - \sigma) - \sum_{r \in x_i^*} c_r(\ell_r(\bar{y}))$. Using that in equilibrium $d_i^* < t_i + \sigma$, we get $\pi_i(\bar{y}) \geq M(t_i - \sigma) - \sum_{r \in R} c_r(T)$, which is positive as $M > 2 \sum_{r \in R} c_r(T) / t_{\min} > \sum_{r \in R} c_r(T) / (t_i - \sigma)$. This is a contradiction to the assumption that $d_i^* = 0$.

Finally, let us assume that $0 < d_i^* < t_i - \sigma$. Referring to Lemma 1, we obtain $U_i'(d_i^*) = \partial (d_i^* \sum_{r \in x_i} c_r(\ell(\bar{x}^*))) / \partial d_i^*$. In particular,

$$M = \sum_{r \in x_i} \left( c_r(\ell(\bar{x}^*)) + d_i^* c_r'(\ell(\bar{x}^*)) \right)$$
$$\leq \sum_{r \in R} \left( c_r(T) + (t_i + \sigma) \max_{\ell \in [t_{\min}, T]} c_r'(\ell) \right),$$

which contradicts the choice of $M$. $\square$

We are now ready to state our first main theorem.

PROPOSITION 1. *Let $\mathcal{M}$ be a congestion model and let $G^w = (N, X, \pi^w)$ be a corresponding weighted congestion game with the vector of demands $(d_i^w)_{i \in N}$. If $G^w$ does not admit a PNE, then there exists a congestion game with variable demands $G(\mathcal{M})$ to the same congestion model, that does not admit a PNE.*

In Harks and Klimm (2010) it is shown that for weighted congestion games, a set $\mathcal{C}$ of (twice continuously differentiable) cost functions is consistent if and only if one of the following cases holds: (i) $\mathcal{C}$ contains only affine functions; (ii) $\mathcal{C}$ contains only exponential functions of type $c(\ell) = a_c e^{\phi \ell} + b_c$ for some constants $a_c. b_c, \phi \in \mathbb{R}$ where $a_c$ and $b_c$ may depend on $c$ while $\phi$ is a common constant for all $c \in \mathcal{C}$. By Proposition 1, we obtain the following result as an immediate corollary.

COROLLARY 1. *If $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ is consistent w.r.t. congestion game with variable demands then one of the following cases holds: (i) $\mathcal{C}$ contains only affine functions; (ii) $\mathcal{C}$ contains only exponential functions.*

The upper result even holds when regarding only network congestion games with variable demands, see the discussion in Harks and Klimm (2010). Because we only need to consider non-negative and strictly increasing cost functions with unbounded marginal costs, for an affine cost function $c$, we can assume that $c(\ell) = a\ell + b$ with $a > 0$, $b \geq 0$. For an

exponential function $c(\ell) = a e^{\phi \ell} + b$ we can assume $a, \phi > 0$ and $b \geq -a$. The function $c$ is called *homogeneously exponential* if $b = 0$ and *inhomogeneously exponential* otherwise. Moreover, when we say that a set $\mathcal{C}$ of functions contains only exponential functions, we assume that there is a *universal* constant $\phi > 0$ such that every $c \in \mathcal{C}$ can be written as $c(\ell) = a_c e^{\phi \ell} + b_c$ for some constants $a_c > 0, b_c \geq -a_c$.

### Excluding Inhomogeneously Exponential Costs.

Next, we show that any inhomogenously exponential function $c(\ell) = a e^{\phi \ell} + b$ with $a, \phi > 0$ and $b \neq 0$ is not consistent w.r.t. congestion games with variable demands. In particular, for every such function $c$ there is a congestion game with variable demands and cost equal to $c$ on all resources that does not admit a PNE. In order to prove this result, we first investigate *congestion games with resource dependent demands*. A complete characterization of the set of cost functions that is consistent w.r.t. this class of games is given. We will use a further refinement of this characterization to show that inhomogenously exponential function are not consistent w.r.t. congestion games with variable demands. Let us first define congestion games with resource dependent demands.

*Definition 1.* Let $\mathcal{M} = (N, F, X, (c_r)_{r \in R})$ be a congestion model and let $(d_{i,r})_{i \in N, r \in R}$ be a matrix of demands. The corresponding *congestion game with resource dependent demands* is the game $G(\mathcal{M}) = (N, X, \pi)$, where $\pi$ is defined as $\pi = \times_{i \in N} \pi_i$, $\pi_i(x) = \sum_{r \in x_i} d_{i,r} c_r(\ell_r(x))$ and $\ell_r(x) = \sum_{j \in N : r \in x_j} d_{j,r}$.

We call a congestion game with resource dependent demands *simple* if each player has a unique demand per strategy, that is, for all $i \in N$ and $x_i \in X_i$ there is $d_{x_i} \in \mathbb{R}_{>0}$ such that $d_{i,r} = d_{x_i}$ for all $r \in x_i$.

Restricting the strategy sets to singletons, we obtain scheduling games on unrelated machines as a special case. Harks et al. (2009) showed that congestion games with resource dependent and affine costs always admit a PNE. Here, we strengthen their result providing a complete characterization of cost functions that are consistent w.r.t. congestion games with resource dependent demand.

THEOREM 1. *A set $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ of cost functions is consistent w.r.t. congestion games with resource dependent demands if and only if $\mathcal{C}$ contains only affine functions.*

Instead of proving this Theorem, we will give a proof of the following technical lemma which strengthens the result of Theorem 1 and which will be the key to provide a complete characterization of the set of cost functions that is consistent w.r.t. congestion games with variable demands.

LEMMA 3. *Let $\Delta > 0$ be arbitrary. Theorem 1 holds even for simple congestion games with resource dependent demands, where for each player one of the following holds: (i) Player $i$ has exactly one strategy, that is, $X_i = \{x_i\}$ with $x_i \subseteq R$; (ii) Player $i$ has exactly two strategies that she uses with demands that differ by $\Delta$, that is, $X_i = \{x_i, \hat{x}\}$ and $|d_{x_i} - d_{\hat{x}_i}| = \Delta$ with $x_i, \hat{x}_i \subseteq R$.*

PROOF. Since the class of games considered in Lemma 3 contains weighted congestion games, it suffices to show that any exponential function is not consistent. To this end let us fix the function $c(\ell) = a e^{\phi \ell} + b$ with $a, \phi \in > 0$ and

$b \geq -a$ and consider for $p, q \in \mathbb{N}_{>0}$ the congestion model $\mathcal{M} = (N, R, X, (c_r)_{r \in R})$ with 5 players and $6q + p$ resources $R = \{r_1, \ldots, r_p, s_1, \ldots, s_{2q}, v_1, \ldots, v_{2q}, w_1, \ldots, w_{2q}\}$. Only the first two players $i \in \{1, 2\}$ have two strategies and will be called *non-trivial* players. Their strategy spaces are $X_1 = \{x_1, \hat{x}_1\}$, $x_1 = \{r_1, \ldots, r_p, s_1, \ldots, s_{2q}\}$, $\hat{x}_1 = \{v_1, \ldots, v_{2q}\}$ and $X_2 = \{x_2, \hat{x}_2\}$, $x_2 = \{r_1, \ldots, r_p, w_1, \ldots, w_{2q}\}$, $\hat{x}_2 = \{s_1, \ldots, s_{2q}\}$. The players 3, 4, and 5 have only one strategy and serve only to increase the costs on some of the resources. They will be called *trivial players*. Their strategies equal $X_3 = \{x_3\} = \{\{r_1, \ldots, r_p\}\}$, $X_4 = \{x_4\} = \{s_1, \ldots, s_{2q}\}$, $X_5 = \{x_5\} = \{\{v_1, \ldots, v_{2q}\}\}$. The strategy spaces are shown in Figure 1. Observe that the strategies of the two non-trivial players are disjoint. For the non-trivial players $i \in \{1, 2\}$, we set $d_{i,r} = 2\frac{\ln 2}{\phi}$ for all $r \in x_i$ and $d_{i,r} = (2 + \frac{p}{q})\frac{\ln 2}{\phi}$ for all $r \in \hat{x}_i$.

Assume that the demand of the non-trivial players 3, 4, and 5 are given by $d_{3,r} = \ln(a_3) / \phi$, $d_{4,r} = \ln(a_4) / \phi$, and $d_{5,r} = \ln(a_5) / \phi$ for all $r \in R$, where $a_3, a_4, a_5 > 1$ are parameters. We will carefully choose the parameters $a_3, a_4$, and $a_5$ such that

$$\gamma = \big((x_1, x_2, x_3, x_4, x_5), (\hat{x}_1, x_2, x_3, x_4, x_5),$$
$$(\hat{x}_1, \hat{x}_2, x_3, x_4, x_5), (x_1, \hat{x}_2, x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5)\big)$$

is an improvement cycle. Calculating the costs of the respective deviating non-trivial player in $\gamma$, we obtain the following necessary and sufficient conditions

$$2^5 \, p \, a_3 + 2^4 \, q \, a_4 > (2 + p/q) \, 2^{3+p/q} \, q \, a_5 \qquad \text{(G1)}$$
$$2^3 \, p \, a_3 + 2^4 \, q > (2 + p/q) \, 2^{3+p/q} \, q \, a_4 \qquad \text{(G2)}$$
$$(2 + p/q) \, 2^{3+p/q} \, q \, a_5 > 2^3 \, p \, a_3 + 2^{6+p/q} \, q \, a_4 \qquad \text{(G3)}$$
$$(2 + p/q) \, 2^{5+p/q} \, q \, a_4 > 2^5 \, p \, a_3 + 2^4 \, q. \qquad \text{(G4)}$$

First, note that the right hand side of inequality (G1) and the left hand side of inequality (G3) are equal. This is due to the fact that the second strategy of the first player can be seen as an outside option of player 1 since none of the resources in that strategy is contained in one of the strategies of the other non-trivial player 2. Thus it is sufficient to consider the inequality

$$2^5 \, p \, a_3 + 2^4 \, q \, a_4 > 2^3 \, p \, a_3 + 2^{6+p/q} \, q \, a_4$$
$$\Leftrightarrow \quad p \, a_3 > \frac{8}{3} \, 2^{p/q} \, q \, a_4 - \frac{2}{3} \, q \, a_4 \qquad \text{(G1')}$$

instead of (G1) and (G3). Once this conditions is satisfied one can choose $a_5$ appropriately. Inequalities (G2) and (G4) give rise to

$$p \, a_3 > 2^{p/q}(2q + p) \, a_4 - 2 \, q, \qquad \text{(G2')}$$
$$2^{p/q}(2q + p) \, a_4 - q/2 > p \, a_3, \qquad \text{(G4')}$$

respectively. It is left to show that inequalities (G1'), (G2'), and (G4') can be satisfied simultaneously. To this end, observe that the function $f(x) = 2^x(x - \frac{2}{3}) + \frac{2}{3}$ is strictly increasing for $x \geq 0$. As $f(0) = 0$, the function $f$ is strictly positive for every $x > 0$. This implies that the inequality $2^{p/q}(\frac{p}{q} - \frac{2}{3}) + \frac{2}{3} > 0$ holds for all $p, q \in \mathbb{N}_{>0}$.

We define $a_4 > \max\{1, 1 / (2^{p/q}(\frac{p}{q} - \frac{2}{3}) + \frac{2}{3}\}$ and $a_3 = 2^{p/q}(2q/p + 1) \, a_4 - \frac{q}{p}$. Since $q > 0$, clearly $a_3$ and $a_4$ satisfy
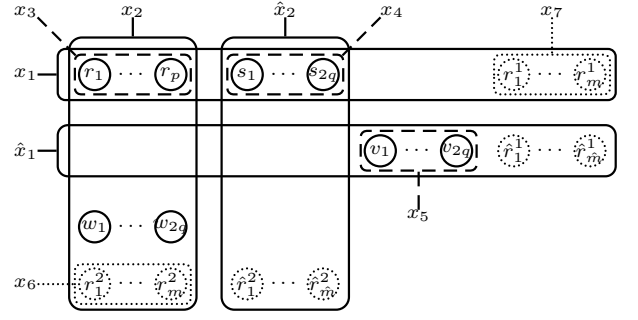


$x_3$    $x_2$    $\hat{x}_2$    $x_4$    $x_7$

**Figure 1: Strategy spaces of the of the congestion games constructed in the proofs of Lemma 3 and Proposition 3. The non-trivial players** 1 **and** 2 **each have two strategies shown by boxes with solid lines while the trivial players** 3, 4 **and** 5 **each have a single strategy drawn as dashed boxes. The resources with dotted circles as well as the trivial players** 6 **and** 7 **are used only in the proof of Proposition 3.**

(G2') and (G4'). As for (G1'), note that

$$p \, a_3 - \frac{8}{3} \, 2^{p/q} \, q \, a_4 + \frac{2}{3} \, q \, a_4$$
$$= a_4 \left(\frac{2}{3} \, q - \frac{2}{3} 2^{p/q} \, p - 2^{p/q} \, p\right) - q > 0.$$

We set

$$a_5 = \frac{2^5 p a_3 + 2^4 q a_4 - 2^3 p a_3 - 2^{6+p/q} q a_4}{2(2 + p/q)2^{3+p/q}q}$$
$$= \frac{1}{(2 + p/q)2^{3+p/q}q}(12 p a_3 + (2^3 - 2^{5+p/q}) q a_4).$$

Clearly, $a_5$ satisfies (G1) and (G3). It is left to show, that $a_3, a_4$, and $a_5$ are not smaller than 1. Since $a_4 > 1$ by definition, (G1') implies $a_3 > \frac{q}{p}(\frac{8}{3}2^{p/q} - \frac{2}{3}) > 1$. As for $a_5$, inequality (G3) gives rise to

$$a_5 > \frac{p \, a_3}{(2 + \frac{p}{q}) \, q \, 2^{p/q}} + \frac{8 \, a_4}{(2 + \frac{p}{q}) \, q}$$
$$= a_4 - \frac{q}{(2 + \frac{p}{q}) \, q \, 2^{p/q}} + \frac{8 \, a_4}{(2 + \frac{p}{q}) \, q} > 1 + \frac{2^{3+p/q} \, q - 1}{(2 + \frac{p}{q}) \, 2^{p/q}} > 1$$

Thus, we have found $a_3, a_4, a_5 > 1$ such that $\gamma$ is an improvement cycle, establishing that the corresponding game does not admit a PNE. $\square$

As a corollary of this result, we obtain that for every non-affine function $c$, there is a simple congestion game with resource dependent demands $G^{\mathrm{rd}} = G^{\mathrm{rd}}(\mathcal{M})$ with costs equal to $c$ on all resources that has an improvement cycle $\gamma = (x^1, x^2, \ldots, x^s, x^1)$. Let $d_i^k$ denote the unique demand that player $i$ puts on all resources in her strategy $x_i^k$ and consider a congestion game with variable demands $G = G(\mathcal{M})$ to the same model, where the utility of each player is constant. Clearly, the cycle

$$\gamma' = \Big(\big(x^1, (d_1^1, \ldots, d_n^1)\big), \big(x^2, (d_1^2, \ldots, d_n^2)\big), \ldots$$
$$\ldots, \big(x^s, (d_1^s, \ldots, d_n^s)\big), \big(x^1, (d_1^1, \ldots, d_n^1)\big)\Big)$$

is also an improvement cycle for the congestion game with variable demand $G$. We have established the following.

PROPOSITION 2. *For a set $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ the following three are equivalent: (i) $\mathcal{C}$ contains only affine functions; (ii) $\mathcal{C}$ is FIP consistent w.r.t. congestion games with variable demands; (iii) $\mathcal{C}$ is consistent w.r.t. congestion games with resource dependent demands.*

Note that for any non-affine cost function $c$, there is a congestion game with variable demands and costs equal to $c$ with an improvement cycle. However, there might be PNE outside that cycle. In the following, we will show that every exponential function is not consistent w.r.t. congestion games with variable demands. To this end, let $c$ be an arbitrary exponential function. Using Lemma 3, there is a simple congestion game with resource dependent demands $G^{rd}$ that does not admit a PNE, see also Figure 1. Our basic idea is to construct a congestion game with variable demands $G$ where players 1 to 5 have access on the same resources. We design the players' concave and non-decreasing utility functions such that for any PNE $(x_i, x^*_{-i}, d^*_i, d^*_{-i})$ in which player $i$ plays configuration $x_i$, the equilibrium demand $d^*_i$ is close to fixed resource dependent demand $d_{x_i}$ as in the proof of Lemma 3. The key to make this idea work is to introduce additional resources (which are drawn with dotted circles in Figure 1) so as to ensure that the players' marginal costs enforce these equilibrium demands. As it turns out, this manipulation can only be done for the case of inhomogenously exponential cost functions. To illustrate this idea, we give an example. If we add two resources with cost $c(\ell) = e^x + 1$ to a players' strategy who plays a demand equal to 1, her costs are increased by $2e + 2$ while her marginal costs are increased by $4e + 2$. If we instead add one resource with the same cost and a trivial player who plays a demand equal to $\ln(2 + e^{-1})$, then her costs are increased by $2e + 2$ as well, but her marginal costs are increased by $4e + 3$. In this fashion, we can increase the marginal costs of one configurations more than in some other configuration while leaving their differences in costs constant. Since increasing the marginal cost of a configuration, decreases the equilibrium demand, the key challenge is to manipulate the players' marginal costs in order to enforce them play the right equilibrium demands.

We are now ready for the main result of this section.

PROPOSITION 3. *Any inhomogenously exponential function is not consistent w.r.t. congestion games with variable demands.*

*Remark 2.* The above result also holds for network congestion games, see the construction in Figure 2.

Below, we can restrict our search space for consistent cost functions to affine functions and homogeneously exponential functions. In the next section, we will show that both classes of functions are consistent.

## 4. CONSISTENT COST FUNCTIONS

### *Affine Functions.*
We will first show below that games with affine cost functions are exact potential games and, thus, they possess a PNE and the $\alpha$-FIP for every $\alpha > 0$.
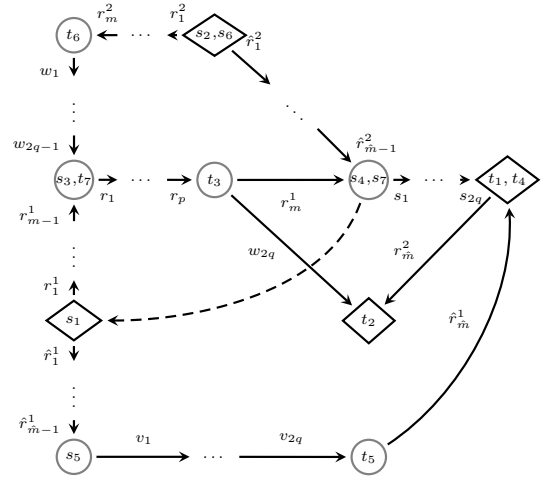


**Figure 2: Network representation of the congestion games constructed in the proof of Proposition 3. Here, resources correspond to the edges of the graph. Each player $i$ is associated with a pair $(s_i, t_i)$ of nodes. The set of strategies of each player equals the set of her directed $(s_i, t_i)$-paths. By adding sufficiently many resources to the path that is indicated by the dashed line, the set of every players undominated non of the non-trivial players will use this path in equilibrium.**

PROPOSITION 4. *Let $G$ be a congestion game with variable demands and affine cost functions. Then, $P(x, d) = \sum_{i \in N} U_i(d_i) - \sum_{r \in x_i} c_r \left( \sum_{j \in \{1,\ldots,i\}: r \in x_j} d_j \right) d_i$ is an exact potential function. Thus, $G$ possesses a PNE and the $\alpha$-FIP for every $\alpha > 0$.*

### *Homogeneously Exponential Cost Functions.*
In the following lemma, we will prove that for every improvement step at least one of the following actions also yields an improvement: (i) adapting the configuration only while leaving the demand constant, (ii) adapting the demand only while leaving the used resources constant. Formally, we call a subset $I' \subseteq I$ of improving moves *essential* if $\{\bar{y} : (\bar{x}, \bar{y}) \in I'\} = \emptyset$ implies $\{\bar{y} : (\bar{x}, \bar{y}) \in I\} = \emptyset$ for all $\bar{x} \in \bar{X}$. Such subsets exist since the set of improving moves $I$ itself is essential. A function $P : \bar{X} \to \mathbb{R}$ is called a *local essential potential* of $G$ if $\bar{x} \in argmax_{\bar{y} \in \bar{X}} P(\bar{y})$ implies $\{\bar{y} : (\bar{x}, \bar{y}) \in I'\} = \emptyset$. Clearly, $\bar{x}$ is a PNE.

LEMMA 4. *Let $G$ be congestion game with variable demands and homogeneously exponential costs. Then, $I' = \left\{ \left((x, d), (y_i, x_{-i}, d)\right) \in I \right\} \cup \left\{ \left((x, d), (x, \Delta_i, d_{-i})\right) \in I \right\}$ is an essential subset of improving moves.*

We will use the above Lemma to prove that every congestion game with variable demands and homogeneously exponential costs admits a local essential potential and thus a PNE.

PROPOSITION 5. *Let $\mathcal{C}$ be a set of homogeneously exponential cost functions. Then, $\mathcal{C}$ is consistent.*

We are now ready to state the main result of this paper.

THEOREM 2. *A set $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ of strictly increasing cost functions is consistent if and only if one of the following cases holds: (i) $\mathcal{C}$ contains only affine functions $c(\ell) = a\ell + b$, $a > 0, b \geq 0$; (ii) $\mathcal{C}$ contains only homogeneously exponential functions $c(\ell) = a\, e^{\phi\ell}$, $a, \phi > 0$, where $\phi$ is a universal constant of all functions in $\mathcal{C}$.*

PROOF. Corollary 1 and Proposition 3 imply the "only if" part. Proposition 4 and Proposition 5 prove the "if" part. $\square$

The characterization of the set of FIP consistent cost functions follows from Proposition 2.

THEOREM 3. *A set $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ of cost functions is FIP consistent if and only if $\mathcal{C}$ contains only affine functions $c(\ell) = a\,\ell + b$ with $a > 0$, $b \geq 0$.*

Our characterizations also hold for network congestion games thus resolving the question raised in. Orda et al. (1993).

# 5. UNIFORM COST MODEL

In the last section, we assumed that the cost function of a resource defines a *per-unit-price* for every player on that resource. Thus, the actual cost for a player is the product of the per-unit-price and the demand of that player. In this section, we will assume that the cost functions on the resources define *uniform* costs (or *per-player costs*), that is, every player pays the same costs regardless of her demand.

Let $\mathcal{M} = (N, R, X, (c_r)_{r \in R})$ be a congestion model and let $(U_i)_{i \in N}$ be a collection of utility functions. We define the corresponding *uniform congestion game with variable demands* as the strategic game $G(\mathcal{M}) = (N, \bar{X}, \pi)$, where $\bar{X} = (X, \mathbb{R}_{\geq 0})$, $\pi = \times_{i \in N} \pi_i$ and $\pi_i(x, d) = U_i(d_i) - \sum_{r \in x_i} c_r(\ell_r(x, d))$, and $\ell_r((x, d)) = \sum_{j \in N : r \in x_j} d_j$.

Most surprisingly, in the uniform cost model, there is a congestion game with variable demands and linear cost functions without PNE (Example 1). This is in stark contrast to Proposition 4 establishing that every congestion game with variable demand in the proportional cost model possesses a PNE. We will shed light on this dichotomy by obtaining a complete characterization.

## Necessary Conditions.

As in the proportional cost model, we can restrict our search space for consistent cost functions to affine and exponential functions. With a slight modification in the proof of Proposition 1 one can establish that every functions that is not consistent w.r.t. weighted congestion games is also non-consistent w.r.t. uniform congestion games with variable demands. Like this, we get the following immediate corollary.

COROLLARY 2. *If $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ is consistent w.r.t. uniform congestion games with variable demands then one of the following cases holds: (i) $\mathcal{C}$ contains only affine functions; (ii) $\mathcal{C}$ contains only exponential functions.*

While the upper result seemingly establishes a structural similarity between the games with proportional costs, it turns out that they behave completely different. As already mentioned, uniform congestion games variable demands and affine costs need not possess a PNE. This holds even for two player games in which the cost function $c_r$ on all resources $r \in R$ equals the identity as illustrated in the following example.

*Example 1.* Regard the model $\mathcal{M} = (N, R, X, (c_r)_{r \in R})$, where $N = \{1, 2\}$ and $R = \{1, \ldots, 46\}$. Each resource has the cost function $c(\ell) = \ell$. We set $X = \times_{i \in N} X_i$, $X_1 = \{\{r_1, \ldots, r_{22}\}, \{r_{23}, \ldots, r_{39}\}\}$, and $X_2 = \{\{r_{23}, \ldots, r_{30}\} \cup \{r_{40}, \ldots, r_{46}\}, \{r_1, \ldots, r_4\} \cup \{r_{31}, \ldots, r_{39}\}\}$. Define $R_{1,2} = \{r_1, \ldots, r_4\}$, $R_{1,0} = \{r_5, \ldots, r_{22}\}$, $R_{2,1} = \{r_{23}, \ldots, r_{30}\}$, $R_{2,2} = \{r_{31}, \ldots, r_{39}\}$, and $R_{0,1} = \{r_{40}, \ldots, r_{46}\}$. The players' strategy spaces are depicted in Figure 3.

For the utility functions $U_1$ and $U_2$ of player 1 and 2, respectively we chose strictly concave, increasing and twice continuously differentiable functions satisfying $U_1(0) = 0$, $U_1(1) = 32$, $U_1(2) = 53$, $U_1'(1) = 22$, $U_1'(2) = 17$ and $U_2(0) = 0$, $U_2(1) = 32$, $U_2(2) = 46$, $U_2'(1) = 15$ $U_2'(2) = 13$, respectively. Clearly, such functions always exists, for instance one can choose polynomials of degree seven to fit the equations above.

Using that the cost functions are equal to the identity, the equilibrium conditions for the Nash equilibrium $(x^*, d^*)$ imply $U_i'(d_i^*) = |x_i|$ for $d_i^* > 0$. Said differently, in any PNE in which player $i$ plays configuration $x_i^*$ and has positive demand $d_i^*$, the equation $d_i^* = U_i'^{-1}(|x_i^*|)$ holds. Note that since the utility functions are strictly concave, the derivatives $U_1'$ and $U_2'$ are convertible and thus their inverse functions $U_1'^{-1}$ and $U_2'^{-1}$ are well-defined. Now, remark that the utility functions $U_i$ of player $i$ is chosen such that in equilibrium $d_i^* \in \{0, 1, 2\}$, since

$$U_1'^{-1}(|R_{1,2} \cup R_{1,0}|) = 1, \qquad U_1'^{-1}(|R_{2,1} \cup R_{2,2}|) = 2,$$
$$U_2'^{-1}(|R_{2,1} \cup R_{0,1}|) = 1, \qquad U_2'^{-1}(|R_{1,2} \cup R_{2,2}|) = 2.$$

Thus, for any equilibrium $(x^*, d^*)$ we get

$$(x_1^*, d_1^*) \in \Big\{ (R_{1,2} \cup R_{1,0}, 1), (R_{2,1} \cup R_{2,2}, 2),$$
$$(R_{1,2} \cup R_{1,0}, 0), (R_{2,1} \cup R_{2,2}, 0) \Big\},$$
$$(x_2^*, d_2^*) \in \Big\{ (R_{2,1} \cup R_{0,1}, 1), (R_{1,2} \cup R_{2,2}, 2),$$
$$(R_{2,1} \cup R_{0,1}, 0), (R_{1,2} \cup R_{2,2}, 0) \Big\}.$$

Remark that the latter two strategies of players 1 and 2 with zero demand are indistinguishable in the sense that they always give the same payoff to every player. First, we will show that there is no PNE in which both players chose a strictly positive demand. We calculate that

$$\pi_1\big((R_{2,1} \cup R_{2,2}, 2), (R_{2,1} \cup R_{0,1}, 1)\big)$$
$$- \pi_1\big((R_{1,2} \cup R_{1,0}, 1), (R_{2,1} \cup R_{0,1}, 1)\big)$$
$$= U_1(2) - 3 \cdot |R_{2,1}| - 2 \cdot |R_{2,2}| - (U_1(1) - 1 \cdot |R_{1,2}| - 1 \cdot |R_{1,0}|)$$
$$= \big(53 - 3 \cdot 8 - 2 \cdot 9\big) - \big(32 - 1 \cdot 4 - 1 \cdot 18\big) = 11 - 10 = 1,$$

and similarly

$$\pi_2\big((R_{2,1} \cup R_{2,2}, 2), (R_{1,2} \cup R_{2,2}, 2)\big)$$
$$- \pi_2\big((R_{2,1} \cup R_{2,2}, 2), (R_{2,1} \cup R_{0,1}, 1)\big) = 2 - 1 = 1,$$
$$\pi_1\big((R_{1,2} \cup R_{1,0}, 1), (R_{1,2} \cup R_{2,2}, 2)\big)$$
$$- \pi_1\big((R_{2,1} \cup R_{2,2}, 2), (R_{1,2} \cup R_{2,2}, 2)\big) = 2 - 1 = 1,$$
$$\pi_2\big((R_{1,2} \cup R_{1,0}, 1), (R_{2,1} \cup R_{0,1}, 1)\big)$$
$$- \pi_2\big((R_{1,2} \cup R_{1,0}, 1), (R_{1,2} \cup R_{2,2}, 2)\big) = 17 - 16 = 1,$$

establishing that neither of these 4 strategy profiles constitutes a PNE. Note that each of these strategy profiles guar-

antees a strictly positive payoff to each player while choosing $d_i^* = 0$ gives a payoff equal to zero. Hence, both players have an incentive to choose a positive demand which together with the observation that $\gamma$ is an improvement cycle establishes that this game does not admit a PNE.
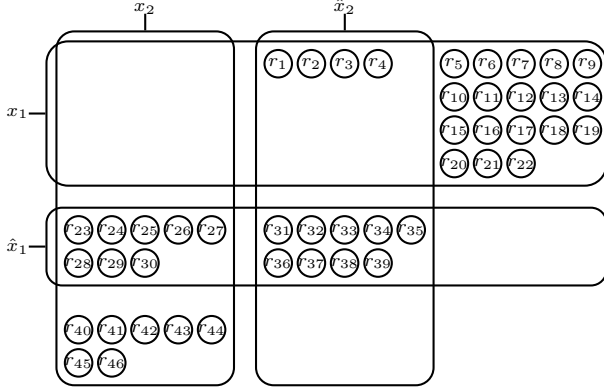


**Figure 3: Strategies of the uniform cost congestion game with variable demands with identity cost and no PNE considered in Example 1.**

Using this example as a blueprint, one can show the following more general result.

PROPOSITION 6. *Any affine function is not consistent w.r.t. uniform congestion games with variable demands.*

In order to show a negative result for inhomogenously exponential functions, we will follow the same line of argumentation as for the proportional cost model. To this end, we first introduce the notion of *uniform congestion games with resource dependent demands*. Let $\mathcal{M} = (N, F, X, (c_r)_{r \in R})$ be a congestion model and let $(d_{i,r})_{i \in N, r \in R}$ be a matrix of facility-dependent demands. The corresponding *uniform congestion game with resource dependent demands* is the strategic game $G(\mathcal{M}) = (N, X, \pi)$, where $\pi = \times_{i \in N} \pi_i$ is defined as $\pi_i(x) = \sum_{r \in x_i} c_r(\ell_r(x))$, $\ell_r(x) = \sum_{j \in N : r \in x_j} d_{j,r}$.

PROPOSITION 7. *Any inhomogenously exponential function is not consistent w.r.t. uniform congestion games with resource dependent demands.*

Using the same idea as in the proof of Proposition 3 the non-consistency of inhomogenously exponential functions carries over to uniform congestion games with variable demands.

PROPOSITION 8. *Any inhomogenously exponential function is not consistent w.r.t. uniform congestion games with variable demands.*

*Sufficient Conditions.*

For uniform congestion games with variable demands and homogenous exponential costs, we establish the existence of a PNE by deriving an *essential* potential function which generalize the potentials considered by Monderer and Shapley Monderer and Shapley (1996). Recall that a subset $I' \subseteq I$ of improving moves is *essential* if $\{\bar{y} : (\bar{x}, \bar{y}) \in I'\} = \emptyset$ implies $\{\bar{y} : (\bar{x}, \bar{y}) \in I\} = \emptyset$ for all $\bar{x} \in \bar{X}$. This definition

motivates the concept of *essential generalized ordinal potentials*. Let $G = (N, \bar{X}, \pi)$ be a strategic game. The function $P : \bar{X} \to \mathbb{R}$ is called an *essential generalized ordinal potential* of $G$ if for all strategy profiles $\bar{x} \in \bar{X}$ there is an essential subset of improving moves $I'$ such that $(\bar{x}, \bar{y}) \in I'$ implies $P(\bar{y}) > P(\bar{x})$. Clearly, a maximizer of $P$ is a PNE.

PROPOSITION 9. *Let $G = (N, \bar{X}, \pi)$ be a uniform congestion game with variable demands and homogenous exponential costs. Then, $G$ admits an essential generalized ordinal potential and possesses a PNE.*

Combining Corollary 2 and Propositions 6, 8, and 9, we state the main result of this section.

THEOREM 4. *A set $\mathcal{C} \subseteq \mathcal{C}^2(\mathbb{R}_{\geq 0})$ of cost functions is consistent w.r.t. uniform congestion games with variable demands if and only if $\mathcal{C}$ contains only homogeneously exponential functions. There is no set of FIP consistent functions.*

## 6. CONCLUSIONS AND OPEN PROBLEMS

We considered the fundamental problem of the existence of PNE and convergence properties of improvement dynamics in congestion games with variable demands. We obtained several characterizations of the cost structure with respect to the existence of PNE and the $\alpha$-FIP. Since our model is general enough to closely capture many elements of practical applications, we are confident that our results help understanding the behavior of myopic play in real systems. We conclude the paper by outlining several research directions that deserve further attention.

While the present work addressed the existence of PNE (and the $\alpha$-FIP) with respect to the cost structure (without constraining the strategy space and the utility functions), it is natural to ask for combinatorial properties of the strategy spaces (such as singletons strategies or strategies given by bases of a matroid) that ensure the existence of PNE for general cost functions. Alternatively, one can restrict the set of feasible utility functions (e.g., assume linear functions) and ask for the existence of PNE. Also the case of symmetry within the set of players (with respect to their utilities, their strategies or both) is open.

Another research direction is to investigate the price of anarchy (stability) in congestion games with variable demands. It would be interesting to characterize the price of anarchy for affine cost functions, because in this case there always exists a PNE.

The design and the analysis of improvement dynamics is fundamental and deserves further investigation. Because our model captures many elements of TCP/IP routing protocols (involving unsplittable routings and elastic demands), it is of practical importance to further investigate the *stability and scalability* of distributed improvement dynamics.

## 7. ACKNOWLEDGMENTS

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