AGM Belief Revision in Dynamic Games*

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ABSTRACT

Within the context of extensive-form (or dynamic) games, we use choice frames to represent the initial beliefs of a player as well as her disposition to change those beliefs when she learns that an information set of hers has been reached. As shown in [5], in order for the revision operation to be consistent with the AGM postulates [1], the player's choice frame must be rationalizable in terms of a total pre-order on the set of histories. We consider four properties of choice frames and show that, together with the hypothesis of a common prior, are necessary and sufficient for the existence of a plausibility order that rationalizes the epistemic state (that is, initial beliefs and disposition to revise those beliefs) of all the players. The plausibility order satisfies the properties introduced in [6] as part of a new definition of perfect Bayesian equilibrium for dynamic games. Thus the present paper provides epistemic foundations for that solution concept.

Categories and Subject Descriptors

G.0 [**Mathematics**]: Miscellaneous—*Game Theory*

General Terms

Game Theory

Keywords

belief revision, plausibility order, perfect Bayesian equilibrium, common prior

1. INTRODUCTION

In a dynamic (or extensive-form) game, a player might find herself having to move at an information set that - according to her prior beliefs - should not have been reached. In such a case the player will have to revise her prior beliefs by formulating a hypothesis about the past moves of other players and a prediction about future moves by herself and the other players. How players revise their beliefs during the play of a game is of central importance in any attempt to provide a solution concept for dynamic games. In [6] we introduced a general notion of perfect Bayesian equilibrium, which can be applied to arbitrary extensive-form games and is intermediate between subgame-perfect equilibrium and sequential equilibrium.[1] In this paper we provide an epistemic foundation of perfect Bayesian equilibrium based on the AGM theory of belief revision proposed by Alchourrón, Gärdenfors and Makinson [1]. We use choice frames to represent the initial beliefs of a player as well as her disposition to change those beliefs when she learns that an information set of hers has been reached. As shown in [5], in order for the revision operation to be consistent with the AGM postulates, the player's choice frame must be rationalizable in terms of a total pre-order on the set of histories. We consider four properties of choice frames that reflect the structure of extensive-form games and show that, together with the hypothesis of a common prior, are necessary and sufficient for the existence of a plausibility order that rationalizes the epistemic state (that is, initial beliefs and disposition to revise those beliefs) of all the players. The plausibility order satisfies the properties introduced in [6] as part of the definition of perfect Bayesian equilibrium.

2. BELIEF REVISION IN GAMES

Choice frames can be used in dynamic or extensive-form games to represent, for every player, her initial beliefs as well as her disposition to change those beliefs when informed that it is her turn to move.

Definition 1. A *choice frame* is a triple $\langle \Omega, \mathcal{E}, f \rangle$ where

- Ω is a non-empty set of *states*. Subsets of Ω are called *events*.
- $\mathcal{E} \subseteq 2^\Omega$ is a collection of events such that $\varnothing \notin \mathcal{E}$ and $\Omega \in \mathcal{E}$.
- $f : \mathcal{E} \to 2^\Omega$ is a function that associates with every event $E \in \mathcal{E}$ an event $f(E)$ satisfying the following properties: (1) $f(E) \subseteq E$ and (2) $f(E) \neq \varnothing$.

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[1] The notion of subgame-perfect equilibrium was introduced by Selten [21], while sequential equilibrium was introduced by Kreps and Wilson [15].

H	set of histories
D	set of decision histories
D_i	set of decision histories of player i
$I_i(h)$	information set of player i that contains $h \in D_i$
$A(h)$	set of actions available at $h \in D$
ha	history that results from adding to h action $a \in A(h)$
$\iota(h)$	player who moves at history $h \in D$

Table 1: Summary of notation

In rational choice theory a set $E \in \mathcal{E}$ is interpreted as a set of available alternatives and $f(E)$ is interpreted as the subset of E which consists of the chosen alternatives (see, for example, [20] and [22]).[2] In our case, we think of the elements of $\mathcal{E}$ as potential items of information and the interpretation of $f(E)$ is that, if informed that event E has occurred, the agent considers as doxastically possible all and only the states in $f(E)$.[3] The set $f(\Omega)$ is interpreted as the set of states that are *initially* considered doxastically possible (that is, before the receipt of information).

Note that in the rational choice literature it is common to impose some structure on the collection of events $\mathcal{E}$ (for example, that it be closed under finite unions or that it be an algebra: see [16, 20, 22]). On the contrary, we allow $\mathcal{E}$ to be an arbitrary subset of 2^{Ω} and typically think of $\mathcal{E}$ as containing only a small number of events. This is characteristically the case in extensive-form games, as shown below.

We make use of the history-based definition of extensive-form game, which is reviewed in Appendix A. For simplicity, we restrict attention to games without chance moves. Table 1 summarizes the notation. For example, in the extensive-form shown in Figure 1, the set of decision histories of player 4 is $D_4 = \{acf, ade, adf, b\}$ and player 4 has two information sets (represented by rounded rectangles): $I_4(acf) = I_4(ade) = \{acf, ade\}$ and $I_4(adf) = I_4(b) = \{adf, b\}$.

Given an extensive form, we associate with every player i a choice frame $\langle \Omega, \mathcal{E}_i, f_i \rangle$ as follows: $\Omega = H$ (recall that H denotes the set of histories), $E \in \mathcal{E}_i$ if and only if either $E = H$ or E consists of an information set of player i together with all the continuation histories, as explained below. Recall that if h is a decision history of player i ($h \in D_i$), player i's information set that contains h is denoted by $I_i(h)$. We shall denote by $\overrightarrow{I_i}(h)$ the set $I_i(h)$ together with the continuation histories:[4]

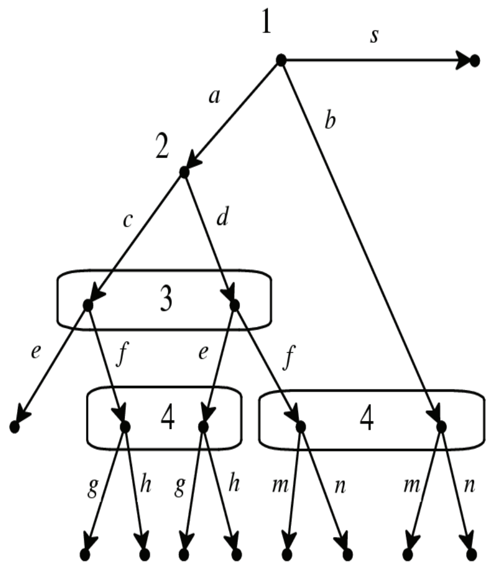

Figure 1: An extensive-form game.

$$\overrightarrow{I_i}(h) = \{x \in H : \exists h' \in I_i(h) \text{ such that } h' \text{ is a prefix of } x\}.$$

For example, in the extensive form of Figure 1, $I_4(b) = \{adf, b\}$ and $\overrightarrow{I_4}(b) = \{adf, b, adfm, adfn, bm, bn\}$.

Thus we define

$$\mathcal{E}_i = \{H\} \cup \{\overrightarrow{I_i}(h) : h \in D_i\}.$$

In the extensive form of Figure 1, $\mathcal{E}_4 = \{H, E, F\}$, where $E = \{acf, ade, acfg, acfh, adeg, adeh\}$ and $F = \{adf, b, adfm, adfn, bm, bn\}$.

Finally, the function f_i provides initial beliefs as well as revised beliefs about past and future moves. For example, in the extensive form of Figure 1 possible beliefs for Player 4 are as follows: $f_4(H) = \{a, ac, ace\}$, $f_4(E) = \{acf, acfh\}$ and $f_4(F) = \{b, bm\}$, where E and F are as given above. The interpretation of this is that Player 4 initially believes that Player 1 will play a, Player 2 will follow with c and Player 3 with e (so that Player 4 does not expect to be asked to make any choices; all this is encoded in $f_4(H)$). If informed that she is at her information set on the left, Player 4 would continue to believe that Player 1 played a and Player 2 followed with c, but she would now believe that Player 3 chose f and she herself plans to choose h (this is encoded in $f_4(E)$). On the other hand, if informed that she is at her information set on the right, Player 4 would believe that Player 1 played b and she herself plans to choose m (this is encoded in $f_4(F)$).

We shall make the following natural assumptions about each player's beliefs. Let $\langle H, \mathcal{E}_i, f_i \rangle$ be the choice frame of player i representing the player's initial beliefs and disposition to change those beliefs. We assume that, for every $E \in \mathcal{E}_i$ and for every $h, h' \in H$,

[2] Choice functions have also been used to provide a semantics for non-monotonic reasoning: see [16]

[3] In order to avoid ambiguity, we use the expression 'doxastically possible' to distinguish between possibility in terms of information (or "objective" possibility) and possibility in terms of beliefs (or "subjective" possibility or "doxastic" possibility). Thus a state ω may be possible according to the information received ($\omega \in E$) but may be ruled out by the agent's beliefs ($\omega \notin f(E)$); the doxastically possible states - when informed that E - are precisely those in $f(E)$. In a framework where beliefs are represented by a probability measure, a state is doxastically possible if and only if it is assigned positive probability.

[4] We call $\overrightarrow{I_i}(h)$ the *augmented* information set of player i at decision history $h \in D_i$. Because of the property of perfect recall (see Appendix A), for every player $i \in N$ and for every $h, h' \in D_i$, either $\overrightarrow{I_i}(h) \cap \overrightarrow{I_i}(h') = \varnothing$ or $\overrightarrow{I_i}(h) \subseteq \overrightarrow{I_i}(h')$ or $\overrightarrow{I_i}(h') \subseteq \overrightarrow{I_i}(h)$. That is, any two different augmented information sets of the same player are either disjoint or one is a subset of the other. Thus if $E, F \in \mathcal{E}_i$ are such that $E \cap F \neq \varnothing$, then either $E \subseteq F$ or $F \subseteq E$. Furthermore, if $h, h' \in D_i$ and h is a prefix of h', then $\overrightarrow{I_i}(h') \subseteq \overrightarrow{I_i}(h)$. Hence, during any play of the game, player i never receives contradictory information; in fact if information F follows information E then $F \subseteq E$, that is, F is a refinement of E.

If $h \in f_i(E)$ and $h' \in E$ is a prefix of h then $h' \in f_i(E)$. (A1)

If $h \in D_i \cap f_i(E)$ then $\exists a \in A(h)$ such that $ha \in f_i(E)$. (A2)

If $h \in D_i$, $h, ha \in f_i(E)$ and $h' \in I_i(h) \cap f_i(E)$ then $h'a \in f_i(E)$. (A3)

Assumption A1 says that the player's beliefs are closed under prefixes: if, when informed that event E has occurred, the player considers history h doxastically possible, and history h' is a prefix of h (thus h' is a necessary condition for h to be reached) then she also considers history h' doxastically possible, as long as h' is compatible with the information received (that is, as long as $h' \in E$).

Assumption A2 says that if h is a decision history of player i ($h \in D_i$), which she considers doxastically possible when informed that E ($h \in f_i(E)$), then she also considers ha doxastically possible for some action a available at h. The interpretation of this is that the player has a belief, that is a plan, about how she would play at h.[5]

Finally, Assumption A3 states that the player's beliefs about her own choices respect her information constraints, in the sense that if she considers histories h and ha doxastically possible (where h is a decision history of hers and a an action available at h) and h' belongs to the same information set as h ($h' \in I_i(h)$), then if she considers h' doxastically possible then she must also consider $h'a$ doxastically possible. The reason for this is that - when taking action a - the player does not know whether she is taking that action at history h or at history h'. Thus if she considers h and h' doxastically possible then she can view ha as doxastically possible if and only if she views $h'a$ as doxastically possible.

Definition 2. The choice frame $\langle H, \mathcal{E}_i, f_i \rangle$ is *rationalizable* if there exists a total pre-order[6] $\precsim_i$ on H such that, $\forall E \in \mathcal{E}_i$, $f_i(E) = \{h \in E : h \precsim_i h', \forall h' \in E\})$.

The interpretation of $h \precsim_i h'$ is that player i judges history h to be at least as plausible as h'. Thus if her epistemic state is captured by a choice frame $\langle H, \mathcal{E}_i, f_i \rangle$ which is rationalizable, then - when she receives information E - player i considers a state doxastically possible if and only if that state is a most plausible state within the set E. An item of information $E \in \mathcal{E}_i$ lists all the histories that are still possible and $f_i(E)$ gives the histories that player i considers most plausible, given that information. Since, by definition of choice frame, $H \in \mathcal{E}_i$, the set $f_i(H)$ gives player i's *initial beliefs*, that is, her beliefs before the game is played, while for $E \in \mathcal{E}_i \backslash \{H\}$, $f_i(E)$ gives player i's *revised beliefs* if informed that E has occurred.

It is shown in [5] that rationalizability of a choice frame is equivalent to compatibility of the associated belief revision policy with the AGM postulates for belief revision [1].[7] [8] [9]

Remark 1. *A choice frame $\langle H, \mathcal{E}_i, f_i \rangle$ of player i contains both (conditional) beliefs about the past and (conditional) beliefs about the player's own future choices. If the choice frame is rationalizable by the total pre-order $\precsim_i$, then - given a decision history h of player i and the corresponding information set $I_i(h)$ - player i's beliefs about past moves are given by the set $\{x \in I_i(h) : x \precsim_i y, \ \forall y \in I_i(h)\}$, that is, the most plausible histories in $I_i(h)$. Furthermore, by Assumptions A2 and A3, for every $h \in D_i$ there exists at least one plausibility preserving action at h.*[10] *The plausibility preserving actions at h represent the beliefs - and thus plans - of player i about her own choice at h.*

Let $\langle H, \mathcal{E}_i, f_i \rangle$ be a choice frame of player i. The following property is known as *Arrow's Axiom* (see, for example, [22], p. 25 and [16], p. 251): $\forall E, F \in \mathcal{E}_i$

$$\text{if } E \subseteq F \text{ and } f_i(F) \cap E \neq \varnothing \text{ then } f_i(E) = f_i(F) \cap E. \quad (AA)$$

For simplicity, we shall restrict attention to extensive forms that satisfy the following condition: $\forall i \in N, \forall h \in D, \forall a \in A(h)$,

$$\text{if } h \in D_i \text{ then } ha \notin D_i. \quad (C)$$

Condition C rules out situations where two consecutive actions are taken by the same player. Thus if, along a possible play of the game, a player takes several actions in a sequence then between any two of them there is an action taken by another - possibly fictitious - player.[11]

[5] The view that "strategies as plans cannot be anything but beliefs of players about their own behavior" is also adopted in [3].

[6] A binary relation $\precsim \ \subseteq H \times H$ is a total pre-order if it is complete ($\forall h, h' \in H$ either $h \precsim h'$ or $h' \precsim h$) and transitive ($\forall h_1, h_2, h_3 \in H$ if $h_1 \precsim h_2$ and $h_2 \precsim h_3$ then $h_1 \precsim h_3$).

[7] Because, typically, the set $\mathcal{E}_i$ of possible items of information contains only few elements, an interpretation of the frame yields only a partial belief revision function. "Compatibility with the AGM postulates"means that the partial belief revision function associated with an arbitrary interpretation of the frame can be extended to a full belief revision function that satisfies the AGM postulates (for details see [5]).

[8] Rationalizable choice frames provide a semantics for qualitative belief revision. A semantics for belief revision in terms of plausibility *measures* is provided in [9, 10].

[9] It should be noted that the AGM theory deals with 'one-stage' belief revision, while in extensive-form games a player might receive information sequentially (when one of her information sets is preceded by another). Thus, in general, in extensive-form games one needs to consider what has been called in the literature 'iterated' belief revision. As noted in Footnote 4, because of the property of perfect recall, if a player receives two sequential pieces of information, E and F, then the latter is a refinement of the former (that is, $F \subseteq E$). In all the theories of iterated belief revision that have been proposed (see, for instance [7, 8, 12, 17]) it is postulated that when information E precedes information F and the latter is a refinement of the former, then the revised beliefs after the sequence $\langle E, F \rangle$ are the same as in the (possibly hypothetical) case where information F is received without it being preceded by E. Our analysis implicitly makes use of this assumption about iterated belief revision.

[10] We say that action $a \in A(h)$ is *plausibility preserving* at h if h is as plausible as ha, that is, if $h \sim_i ha$, where $h \sim_i ha$ is a short-hand for $h \precsim_i ha$ and $ha \precsim_i h$.

[11] If an extensive form does not satisfy Condition C then one can transform it into one that does, by adding a fictitious player between two consecutive actions of the same player and assigning to the fictitious player only one action. Such a trasformation would be "inessential" in the sense that, for example, it would not affect the set of sequential equilibria.

The proofs of the following propositions are given in Appendix B.

Proposition 1. *Fix an extensive form that satisfies Condition C. Let* $\langle H, \mathcal{E}_i, f_i \rangle$ *be a choice frame representing player i's initial beliefs and disposition to change those beliefs. Then the following are equivalent:*

(a) $\langle H, \mathcal{E}_i, f_i \rangle$ *satisfies Arrow's Axiom and Assumptions A1-A3.*

(b) There is a total pre-order $\precsim_i$ *that rationalizes* $\langle H, \mathcal{E}_i, f_i \rangle$ *and satisfies the following properties:*

PL1. $\forall h \in D,\ \forall a \in A(h),\quad h \precsim_i ha.$

PL2i. $\forall h \in D_i$, *(1)* $\exists a \in A(h)$ *such that* $ha \precsim_i h$ *and, (2)* $\forall a \in A(h)$, *if* $ha \precsim_i h$ *then* $h'a \precsim_i h'$, $\forall h' \in I_i(h)$.

Let $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ be a profile of choice frames representing the initial beliefs and disposition to revise those beliefs of all the players. Let $\mathcal{P}_i$ be the set of total pre-orders that rationalize $\langle H, \mathcal{E}_i, f_i \rangle$ and satisfy Properties PL1 and PL2i (Proposition 1 above gives necessary and sufficient conditions for $\mathcal{P}_i \neq \varnothing$).

Definition 3. We say that the profile $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ *admits a common prior* if $\bigcap_{i \in N} \mathcal{P}_i \neq \varnothing$, that is, if there exists a total pre-order $\precsim$ on H that rationalizes the beliefs of all the players[12] and satisfies Properties PL1 and PL2i for every $i \in N$. We call any element of $\bigcap_{i \in N} \mathcal{P}_i$ a *common prior.*[13]

If the players have a common prior then they share the same initial beliefs and the same disposition to change those beliefs in response to the same information. However, the existence of a common prior is consistent with the players holding different beliefs during any particular play of the game, since they will typically receive different information. A common prior can also be viewed as representing the initial beliefs and belief revision policy of an external observer (the external-observer point of view is pursued in [13]).

Definition 4. A total pre-order $\precsim$ on the set of histories H is called a *plausibility order* if it satisfies the following properties:

PL1. $\forall h \in D,\ \forall a \in A(h),\quad h \precsim ha.$

PL2. $\forall h \in D$, (1) $\exists a \in A(h)$ such that $ha \precsim h$ and, (2) $\forall a \in A(h)$, if $ha \precsim h$ then $h'a \precsim h'$, $\forall h' \in I_{\iota(h)}(h)$, where $\iota(h)$ is the player who moves at h.

Proposition 2. *Fix an extensive form that satisfies Condition C. Let* $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ *be a profile of choice frames representing the initial beliefs and disposition to revise those beliefs of all the players. If the choice frame of each player satisfies Arrow's Axiom and Assumptions (A1)-(A3) and the profile* $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ *admits a common prior then every common prior is a plausibility order.*

[12]That is, $\forall i \in N,\ \forall E \in \mathcal{E}_i,\ f_i(E) = \{h \in E : h \precsim h',\ \forall h' \in E\}$.

[13]There may be several total pre-orders that play the role of a common prior, but they all yield the same conditional beliefs, given the possible items of information encoded in the extensive form.

3. CHOICE FRAMES AND ASSESSMENTS

Solution concepts for extensive-form games that go beyond subgame-perfect equilibrium (such as sequential equilibrium) are defined in terms of assessments. The notion of assessment is reviewed in detail in Appendix A. An assessment is a pair (σ, μ) where σ is a (behavior) strategy profile (that is, an n-tuple of strategies, one for each player) and μ is a list of probability distributions, one for each information set, over the histories that constitute that information set. In [6] an assessment is defined to be AGM-consistent if there is a plausibility order (see Definition 4 above) that rationalizes (σ, μ) in the sense that the actions that are played with positive probability coincide with the plausibility-preserving actions and the histories that are assigned positive probability are those that are most plausible within each information set. The formal definition is as follows (where $\sigma(a)$ denotes the probability with which action a is chosen according to σ and $\mu(h)$ is the probability assigned to history h by the relevant part of μ).

Definition 5. An assessment (σ, μ) is *AGM-consistent* if there exists a plausibility order $\precsim$ on H such that:

(1) $\sigma(a) > 0$ if and only if $h \sim ha$, and

(2) $\mu(h) > 0$ if and only if $h \precsim h', \forall h' \in I_{\iota(h)}(h)$.

Let $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ be a profile of choice frames such that (i) the choice frame of each player satisfies Arrow's Axiom and Assumptions (A1)-(A3) and (2) the profile $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ admits a common prior. Let $\precsim$ be any common prior. By Proposition 2, $\precsim$ is a plausibility order. Corresponding to $\precsim$ there will be many AGM-consistent assessments (σ, μ), all of which share the same support (for σ the support is given by the plausibility-preserving actions and for μ the support is given by the most plausible histories within each information set). [14] It can be shown that the converse is also true, that is, given an AGM-consistent assessment (σ, μ) one can extract from it a profile $\{\langle H, \mathcal{E}_i, f_i \rangle\}_{i \in N}$ of choice frames that satisfies the hypothesis of Proposition 2. Thus the analysis of this paper provides a foundation for the notion of AGM-consistent assessment in terms of epistemic states for the players that satisfy the AGM postulates for belief revision.

4. CONCLUSION

As shown in [6], the qualitative notion of AGM-consistency of assessments (Definition 5) is a generalization of the notion of consistency proposed by Kreps and Wilson [15] as part of the definition of sequential equilibrium. The conceptual content of the notion of Kreps-Wilson consistency is not clear and several attempts have been made to clarify it by relating it to more intuitive notions, such as 'structural consistency' ([14]), 'generally reasonable extended assessment' ([11]), 'stochastic independence' ([2, 13]).[15] In this paper we introduced a representation of the epistemic state of players in dynamic games based on of choice frames, which provide a link to the AGM theory of belief revision [1, 5]. We have identified four properties of individual frames that, together with the hypothesis of a common prior, are equivalent to the existence of an AGM-consistent assessment.

[14]The definition of perfect Bayesian equilibrium put forward in [6] specifies a way in which the probabilities can be chosen on these supports so as to make μ compatible with σ and Bayes' rule.

[15]Perea *et al* [19] offer an algebraic characterization of consistent assessments.

APPENDIX

A. EXTENSIVE FORMS AND ASSESSMENTS

In this appendix we review the history-based definition of extensive-form game (see, for example, [18]). If A is a set, we denote by A^* the set of finite sequences in A. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $1 \leq j \leq k$, the sequence $h' = \langle a_1, ..., a_j \rangle$ is called a prefix of h.[16] If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $a \in A$, we denote the sequence $\langle a_1, ..., a_k, a \rangle \in A^*$ by ha.

A finite *extensive form* without chance moves is a tuple $\left\langle A, H, N, \iota, \{\approx_i\}_{i \in N} \right\rangle$ whose elements are:

- A finite set of actions A.
- A finite set of histories $H \subseteq A^*$ which is closed under prefixes (that is, if $h \in H$ and $h' \in A^*$ is a prefix of h, then $h' \in H$). The null history $\langle\rangle$, denoted by $\emptyset$, is an element of H and is a prefix of every history. A history $h \in H$ such that, for every $a \in A$, $ha \notin H$, is called a terminal history. The set of terminal histories is denoted by Z. Let $D = H \setminus Z$ denote the set of non-terminal or decision histories. For every history $h \in H$, we denote by $A(h)$ the set of actions available at h, that is, $A(h) = \{a \in A : ha \in H\}$. Thus $A(h) \neq \varnothing$ if and only if $h \in D$. We assume that $A = \bigcup_{h \in D} A(h)$ (that is, we restrict attention to actions that are available at some decision history).
- A finite set $N = \{1, ..., n\}$ of players.
- A function $\iota : D \to N$ that assigns a player to each decision history; thus $\iota(h)$ is the player who moves at history h. For every $i \in N$, let $D_i = \iota^{-1}(i)$ be the histories assigned to player i. Thus $\{D_1, ..., D_n\}$ is a partition of D.
- For every player $i \in N$, $\approx_i$ is an equivalence relation on D_i. The interpretation of $h \approx_i h'$ is that, when choosing an action at history $h \in D_i$, player i does not know whether she is moving at h or at h'. The equivalence class of $h \in D_i$ is denoted by $I_i(h)$ and is called an information set of player i; thus $I_i(h) = \{h' \in D_i : h \approx_i h'\}$. The following restriction applies: if $h' \in I_i(h)$ then $A(h') = A(h)$, that is, the set of actions available to a player is the same at any two histories that belong to the same information set of that player.
- The following property, known as *perfect recall*, is assumed: for every player $i \in N$, if $h_1, h_2 \in D_i$, $a \in A(h_1)$ and $h_1 a$ is a prefix of h_2 then for every $h' \in I_i(h_2)$ there exists an $h \in I_i(h_1)$ such that ha is a prefix of h'. Intuitively, perfect recall requires a player to remember what she knew in the past and what actions she took previously (see [4]).

In order to simplify the notation in the proofs, we shall assume that no action is available at more than one information set: $\forall h, h' \in D$, if $A(h) \cap A(h') \neq \varnothing$ then $h' \in I_{\iota(h)}(h)$.

Given an extensive form, one obtains a *game based on it* by adding, for every player $i \in N$, a utility (or payoff) function $U_i : Z \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers; recall that Z is the set of terminal histories).

[16] In particular, every history is a prefix of itself.

Given an extensive form, a *pure strategy* of player $i \in N$ is a function that associates with every information set of player i an action at that information set, that is, a function $s_i : D_i \to A$ such that (1) $s_i(h) \in A(h)$ and (2) if $h' \in I_i(h)$ then $s_i(h') = s_i(h)$. A *behavior strategy* of player i is a collection of probability distributions, one for each information set, over the actions available at that information set; that is, a function $\sigma_i : D_i \to \Delta(A)$ (where $\Delta(A)$ denotes the set of probability distributions over A) such that (1) $\sigma_i(h)$ is a probability distribution over $A(h)$ and (2) if $h' \in I_i(h)$ then $\sigma_i(h') = \sigma_i(h)$. Note that a pure strategy is a special case of a behavior strategy where each probability distribution is degenerate. A *behavior-strategy profile* is an n-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ where, for every $i \in N$, σ_i is a behavior strategy of player i. Given our assumption that no action is available at more than one information set, without risking ambiguity we shall denote by $\sigma(a)$ the probability assigned to action a by the relevant component of the strategy profile σ.

A *system of beliefs*, is a collection of probability distributions, one for every information set, over the elements of that information set, that is, a function $\mu : D \to \Delta(H)$ such that (1) if $h \in D_i$ then $\mu(h)$ is a probability distribution over $I_i(h)$ and (2) if $h \in D_i$ and $h' \in I_i(h)$ then $\mu(h) = \mu(h')$. Without risking ambiguity we shall denote by $\mu(h)$ the probability assigned to history h by the system of beliefs μ.[17]

An *assessment* is a pair (σ, μ) where σ is a behavior-strategy profile and μ is a system of beliefs.

B. PROOFS

The proof of Proposition 1 requires several preliminary results. The idea of the proof is to construct a binary relation on the set of histories H that satisfies Properties PL1 and PL2i and extend it to a total pre-order which is then shown to rationalize the given choice frame. The extension is obtained by invoking Proposition 3 below, which is known as Szpilrajn's theorem (for a proof see [22], p. 14). First we give the definition of extension. Given a binary relation R on H (thus $R \subseteq H \times H$) we shall interchangeably use the notation hRh' and $(h, h') \in R$.

Definition 6. Let R be a binary relation on H and $\precsim$ a total pre-order on H. We say that $\precsim$ *extends* R if
(1) if $(h, h') \in R$ then $(h, h') \in \precsim$ and
(2) if $(h, h') \in R$ and $(h', h) \notin R$ then $(h', h) \notin \precsim$.

Proposition 3. *(Szpilrajn's theorem) Let R be a binary relation on H which is reflexive and transitive. Then there exists a total pre-order $\precsim$ on H which extends R.*

The following proposition is more general than Proposition 1 in that it applies to arbitrary extensive forms (that is, Condition C is *not* assumed), but it is also weaker since it only refers to Property PL1 and Assumption A1. The proof illustrates the strategy used in proving Proposition 1.

[17] A more precise notation would be $\mu(h)(h)$: if $h \in D_i$ then $\mu(h)$ is a probability distribution over $I_i(h)$ and, for every $h' \in I(h)$, $\mu(h) = \mu(h')$ so that $\mu(h)(h) = \mu(h')(h)$. With slight abuse of notation we denote this common probability by $\mu(h)$.

Proposition 4. *Let $\langle H, \mathcal{E}_i, f_i\rangle$ be a choice frame of player i. The following are equivalent:*

(a) *There is a total pre-order $\precsim$ on H that satisfies property PL1 ($\forall h \in D$, $\forall a \in A(h)$, $h \precsim ha$) and rationalizes $\langle H, \mathcal{E}_i, f_i\rangle$ ($\forall E \in \mathcal{E}_i$, $f_i(E) = \{h \in E : h \precsim h', \forall h' \in E\}$),*

(b) *$\langle H, \mathcal{E}_i, f_i\rangle$ satisfies Arrow's Axiom and Assumption A1.*

Proof. (a) $\Rightarrow$ (b). Let $\precsim$ be a total pre-order on H that satisfies property PL1 and is such that

$$\forall E \in \mathcal{E}_i, \quad f_i(E) = \{h \in E : h \precsim h', \forall h' \in E\}. \qquad (1)$$

First we show that Arrow's Axiom (AA) holds. Let $F, G \in \mathcal{E}_i$ be such that $F \subseteq G$ and $f_i(G) \cap F \neq \varnothing$. We need to show that $f_i(F) = f_i(G) \cap F$. Fix an arbitrary $h \in f_i(G) \cap F$. By (1), $h \precsim h', \forall h' \in G$ and thus, since $F \subseteq G$, $h \precsim h', \forall h' \in F$. Hence, by (1) and the fact that $h \in F$, $h \in f_i(F)$. Conversely, fix an arbitrary $h \in f_i(F)$. Then, by (1),

$$h \precsim h', \forall h' \in F. \qquad (2)$$

By hypothesis, $f_i(G) \cap F \neq \varnothing$. Fix an arbitrary $h_0 \in f_i(G) \cap F$. Since $h_0 \in f_i(G)$, by (1), $h_0 \precsim h', \forall h' \in G$. Since $h_0 \in F$, by (2) $h \precsim h_0$. Thus, by transitivity of $\precsim$, $h \precsim h', \forall h' \in G$, so that, by (1), $h \in f_i(G)$ (note that $h \in G$ since $h \in f_i(F) \subseteq F$ and $F \subseteq G$). Hence $h \in f_i(G) \cap F$.

Next we prove that Assumption A1 is satisfied. Fix arbitrary $E \in \mathcal{E}_i$ and $h \in f_i(E)$. Let $h' \in E$ be a prefix of h. We need to show that $h' \in f_i(E)$. By (1) (since $h \in f_i(E)$), $h \precsim y, \forall y \in E$. By Property PL1 and transitivity of $\precsim$, $h' \precsim h$.[18] Thus, by transitivity of $\precsim$, $h' \precsim y, \forall y \in E$, so that, by (1), $h' \in f_i(E)$.

(b) $\Rightarrow$ (a). Let $\langle H, \mathcal{E}_i, f_i\rangle$ satisfy Arrow's Axiom and Assumption A1. Define the following binary relation S on H:

$$(h, h') \in S \text{ if and only if } \begin{cases} \text{either} & \text{(a)} \quad h \text{ is a prefix of } h' \\ \text{or} & \text{(b)} \quad \exists h_1 \in H, \exists E \in \mathcal{E}_i : \\ & \qquad h \text{ is a prefix of } h_1, \\ & \qquad h_1 \in f_i(E) \\ & \qquad \text{and } h' \in E. \end{cases} \qquad (3)$$

First we show that S is reflexive and transitive. Reflexivity follows from (a) of (3) and the fact that, by definition of prefix, every history is a prefix of itself. To prove transitivity, fix arbitrary $h, h', h'' \in H$ and suppose that hSh' and $h'Sh''$. We need to show that hSh''. If h is a prefix of h' and h' is a prefix of h'', then h is a prefix of h'' and thus hSh''. If h is a prefix of h' while h' is not a prefix of h'', then $\exists h_1 \in H$, $\exists E \in \mathcal{E}_i$ such that h' is a prefix of h_1, $h_1 \in f_i(E)$ and $h'' \in E$. Then (since h is a prefix of h' and h' is a prefix of h_1) h is a prefix of h_1 and thus hSh'' by (b) of (3). If h is not a prefix of h' while h' is a prefix of h'', then $\exists h_1 \in H$, $\exists E \in \mathcal{E}_i$ such that h is a prefix of h_1, $h_1 \in f_i(E)$ and $h' \in E$. Then, since $h' \in E$ and h' is a prefix of h'', $h'' \in E$ (this follows from the definition of $\mathcal{E}_i$). Thus hSh'' by (b) of (3). We are left with the case where h is not a prefix of h' and h' is not a prefix of h''. Then $\exists x_1, y_1 \in H$, $\exists E, F \in \mathcal{E}_i$ such that (i) h is a prefix of x_1, (ii) $x_1 \in f_i(E)$, (iii) $h' \in E$, (iv) h' is a prefix of y_1, (v) $y_1 \in f_i(F)$ and (vi) $h'' \in F$. By (iii) and (iv) $y_1 \in E$. Hence, by (v) (since $f_i(F) \subseteq F$), $E \cap F \neq \varnothing$ so that either $F \subseteq E$ or $E \subseteq F$ (see Footnote 4). Consider first the case where $F \subseteq E$. Then, since $h'' \in F$, we have that $h'' \in E$. By (b) of (3), it follows from this, (i) and (ii) that hSh''. Now consider the case where $E \subseteq F$. Since $y_1 \in f_i(F)$ and $y_1 \in E$, $f_i(F) \cap E \neq \varnothing$. Thus, by Arrow's Axiom, $f_i(E) = f_i(F) \cap E$. Hence, since $x_1 \in f_i(E)$, $x_1 \in f_i(F)$. Thus, since since h is a prefix of x_1, $x_1 \in f_i(F)$ and $h'' \in F$, by (b) of (3) hSh''.

Since S is reflexive and transitive, by Proposition 3, there exists a total pre-order $\precsim$ on H which extends S (see Definition 6). Fix an arbitrary such total pre-order $\precsim$. We want to show that $\precsim$ satisfies Property PL1 and rationalizes $\langle H, \mathcal{E}_i, f_i\rangle$. Since, for every $h \in D$ and $a \in A(h)$, h is a prefix of ha, $(h, ha) \in S$ and thus, since S is a subset of $\precsim$, $h \precsim ha$ so that $\precsim$ satisfies Property PL1. Now fix an arbitrary $E \in \mathcal{E}_i$. We need to show that $f_i(E) = \{h \in E : h \precsim h', \forall h' \in E\}$. Fix arbitrary $h \in f_i(E)$ and $h' \in E$. Then (since h is a prefix of itself) by (b) of (3) hSh' and thus, since S is a subset of $\precsim$, $h \precsim h'$. Hence $f_i(E) \subseteq \{h \in E : h \precsim h', \forall h' \in E\}$. For the converse, let $h \in E$ be such that $h \precsim h'$ for all $h' \in E$; we need to show that $h \in f_i(E)$. Fix an arbitrary $h_0 \in f_i(E)$ (recall that, by definition of choice frame, $f_i(E) \neq \varnothing$). If h is a prefix of h_0 then, by Assumption A1, $h \in f_i(E)$. Suppose that h is not a prefix of h_0. By definition of S (since $h \in E$ and $h_0 \in f_i(E)$ and h_0 is a prefix of itself), $(h_0, h) \in S$. If $(h, h_0) \notin S$, then, since $\precsim$ is an extension of S (see Definition 6), $(h, h_0) \notin \precsim$, contradicting our hypothesis that $h \precsim h', \forall h' \in E$. Thus it must be that $(h, h_0) \in S$. Then (since h is not a prefix of h_0) there exist $h_1 \in H$ and $F \in \mathcal{E}_i$ such that (i) h is a prefix of h_1, (ii) $h_1 \in f_i(F)$ and (iii) $h_0 \in F$. Then (since $h_0 \in F$ and $h_0 \in f_i(E) \subseteq E$), $E \cap F \neq \varnothing$ and thus (see Footnote 4) either $E \subseteq F$ or $F \subseteq E$ (see Footnote 4). Suppose first that $E \subseteq F$. Since $h \in E$ and h is a prefix of h_1, $h_1 \in E$. Thus, since $h_1 \in f_i(F)$, $f_i(F) \cap E \neq \varnothing$ and, by Arrow's Axiom, $f_i(E) = f_i(F) \cap E$. Hence $h_1 \in f_i(E)$ and thus, by Assumption A1 (since h is a prefix of h_1 and $h \in E$), $h \in f_i(E)$. Suppose now that $F \subseteq E$. Then, since $h_0 \in f_i(E) \cap F$, $f_i(E) \cap F \neq \varnothing$ and thus, by Arrow's Axiom, $f_i(F) = f_i(E) \cap F$. Thus, since $h_1 \in f_i(F)$, $h_1 \in f_i(E)$ and therefore, by Assumption A1 (since h is a prefix of h_1), $h \in f_i(E)$. $\square$

The proof of Proposition 1 follows the same strategy, starting from a relation that satisfies also Property PL2i. In order to do this we need several preliminary lemmas. Note that Condition C is used only in the proof of Lemma 3 and is not needed for any other result.

Lemma 1. *Fix an arbitrary choice frame $\langle H, \mathcal{E}_i, f_i\rangle$ of player i. Let $h \in D_i$ be a decision history of player i and let $F \in \mathcal{E}_i$ be such that $h \in F$. Then $F \supseteq E$, where $E = \overrightarrow{I_i}(h) \in \mathcal{E}_i$.*

Proof. Since $h \in F \in \mathcal{E}_i$, there exists an $x \in D_i$ such that x is a prefix of h and $F = \overrightarrow{I_i}(x)$. If $x = h$ then $F = E$. If $x \neq h$, then, by perfect recall, every $h' \in I_i(h)$ has a prefix in $I_i(x)$ and thus $E = \overrightarrow{I_i}(h) \subseteq \overrightarrow{I_i}(x) = F$. $\square$

[18]Since h' is a prefix of h, there exist $a_1, ..., a_m \in A$ ($m \geq 0$) such that $h = h'a_1...a_m$. By PL1 $h' \precsim h'a_1 \precsim h'a_1a_2 \precsim ... \precsim h'a_1...a_m = h$. Thus, by transitivity of $\precsim$, $h' \precsim h$.

Fix an arbitrary choice frame $\langle H, \mathcal{E}_i, f_i\rangle$ of player i. Define the following binary relations on H:

$$(x,y) \in R_1 \text{ if and only if } \begin{cases} x \in D_i,\ y \in I_i(x) \text{ and} \\ x \in f_i(E) \text{ where } E = \overrightarrow{I_i}(x). \end{cases} \tag{4}$$

$$(x,y) \in R_2 \text{ if and only if } \begin{cases} y \in D_i,\ x = ya \text{ for some } a \in A(y) \\ \text{and } \exists h \in I_i(y) \text{ such that} \\ h, ha \in f_i(E) \text{ where } E = \overrightarrow{I_i}(y) \end{cases} \tag{5}$$

$$(x,y) \in R_3 \text{ if and only if } x \in f_i(H) \text{ and } y \text{ is a prefix of } x. \tag{6}$$

$$(x,y) \in R_4 \text{ if and only if } \begin{cases} y \in D_i,\ y \text{ is a prefix of } x \text{ and} \\ x \in f_i(E) \text{ where } E = \overrightarrow{I_i}(y). \end{cases} \tag{7}$$

$$(x,y) \in R_5 \text{ if and only if } x \text{ is a prefix of } y. \tag{8}$$

$$R = \bigcup_{j=1}^{5} R_j. \tag{9}$$

$$R^* = \text{ transitive closure of } R. \tag{10}$$

Remark 2. *The relations R_1 and R_5 are transitive. Furthermore, R_5 is reflexive (since every history is a prefix of itself).*

Remark 3. *Let $\langle H, \mathcal{E}_i, f_i\rangle$ be a choice frame of player i. If $h \in D_i$ and $a \in A(h)$ are such that $(ha,h) \in R_2$ then $(h'a,h') \in R_2$, for every $h' \in I_i(h)$.*[19]

Lemma 2. *Let $\langle H, \mathcal{E}_i, f_i\rangle$ be a choice frame of player i that satisfies Arrow's Axiom and Assumption A1. Let $h \in D_i$ and $a \in A(h)$ be such that $(ha,h) \notin R_2$. Let $\langle x_1, ..., x_m\rangle$ ($m \geq 2$) be a sequence in H such that $x_1 = ha$ and, $\forall j = 1, ..., m-1$, $(x_j, x_{j+1}) \in R$ (where R is give by (9)). Then, $\forall j = 1, ..., m$, $\exists h_j \in I_i(h)$ such that $h_j a$ is a prefix of x_j.*

Proof. This is clearly true for $j = 1$ (take $h_1 = h$). We now show that if the statement is true for $j \geq 1$ then it is true for $j+1$. Let $h_j \in H$ be such that

$$h_j \in I_i(h) \text{ and } h_j a \text{ is a prefix of } x_j. \tag{11}$$

By hypothesis, $(x_j, x_{j+1}) \in R$. We need to consider all the possible cases.

Case 1: $(x_j, x_{j+1}) \in R_5$. Then x_j is a prefix of x_{j+1} and thus, since $h_j a$ is a prefix of x_j, $h_j a$ is a prefix of x_{j+1}.

Case 2: $(x_j, x_{j+1}) \in R_4$. Then $x_{j+1} \in D_i$, x_{j+1} is a prefix of x_j and

$$x_j \in f_i(F) \text{ where } F = \overrightarrow{I_i}(x_{j+1}). \tag{12}$$

Since both $h_j a$ and x_{j+1} are prefixes of x_j, either $h_j a$ is a prefix of x_{j+1} (with $x_{j+1} = h_j a$ as a special case), and thus the claim is true (take $h_{j+1} = h_j$), or x_{j+1} is a prefix of $h_j a$ and $x_{j+1} \neq h_j a$. Consider the latter case; then x_{j+1} is a prefix of h_j. Let $E = \overrightarrow{I_i}(h) = \overrightarrow{I_i}(h_j)$. Then, by perfect recall (since $x_{j+1} \in D_i$), $E \subseteq F$. Thus, since, by (11) and (12), $x_j \in f_i(F) \cap E$, by Arrow's Axiom $f_i(E) = f_i(F) \cap E$ so that $x_j \in f_i(E)$. Hence, by Assumption A1, since $h_j a$ is a prefix of x_j, $h_j a \in f_i(E)$ and thus also $h_j \in f_i(E)$; but this implies, by definition of R_2 (see (5)), that $(ha, h) \in R_2$, contrary to our hypothesis. Thus if $(x_j, x_{j+1}) \in R_4$ then $h_j a$ is a prefix of x_{j+1}.

Case 3: we show that it cannot be that $(x_j, x_{j+1}) \in R_3$. In fact, $(x_j, x_{j+1}) \in R_3$ requires that $x_j \in f_i(H)$ so that, by Arrow's Axiom, $f_i(E) = f_i(H) \cap E$ (where $E = \overrightarrow{I_i}(h)$; note that $x_j \in E$). Hence $x_j \in f_i(E)$ and, by Assumption A1 (since $h_j a$ is a prefix of x_j), $h_j a \in f_i(E)$ an thus also $h_j \in f_i(E)$; but this implies, by definition of R_2, that $(ha, h) \in R_2$, contrary to our hypothesis.

Case 4: $(x_j, x_{j+1}) \in R_2$. Then, by definition of R_2, either (i) $x_{j+1} = h_j$ (if $x_j = h_j a$) or (ii) $x_{j+1} = h_j a b_1 ... b_{m-1}$ (if $x_j = h_j a b_1 ... b_m$ for some $b_1, ..., b_m \in A$, $m \geq 1$). In case (i), by definition of R_2, $\exists h_0 \in I_i(h_j)$ such that $h_0, h_0 a \in f_i(E)$ (where $E = \overrightarrow{I_i}(h_j)$). Since $I_i(h_j) = I_i(h)$, it would follow that $(ha, h) \in R_2$, contradicting our hypothesis. In case (ii) $h_j a$ is a prefix of x_{j+1}.

Case 5: $(x_j, x_{j+1}) \in R_1$. Then $x_j \in D_i$ and $x_{j+1} \in I_i(x_j)$. By perfect recall, since $h_j a$ is a prefix of x_j, $\exists h' \in I_i(h_j) = I_i(h)$ such that $h'a$ is a prefix of x_{j+1}. □

Corollary 1. *Let $\langle H, \mathcal{E}_i, f_i\rangle$ be a choice frame of player i that satisfies Arrow's Axiom and Assumption A1. Let $h \in D_i$ and $a \in A(h)$. Then $(ha, h) \in R^*$ if and only if $(ha, h) \in R_2$.*

Proof. If $(ha, h) \in R_2$ then, since $R_2 \subseteq R \subseteq R^*$, $(ha, h) \in R^*$. To prove the converse, suppose that $(ha, h) \in R^*$. Then there exists a sequence $\langle x_1, ..., x_m\rangle$ ($m \geq 2$) in H such that $x_1 = ha$, $x_m = h$ and, $\forall j = 1, ..., m-1$, $(x_j, x_{j+1}) \in R$. If $(ha, h) \notin R_2$ then, by Lemma 2, $\forall j = 1, ..., m$, $\exists h_j \in I_i(h)$ such that $h_j a$ is prefix of x_j. In particular, $\exists h_m \in I_i(h)$ such that $h_m a$ is prefix of $x_m = h$, but this violates perfect recall. □

Lemma 3. *Fix an extensive form that satisfies Condition C. Let $\langle H, \mathcal{E}_i, f_i\rangle$ be a choice frame of player i that satisfies Arrow's Axiom and Assumptions A1 and A3. Let $F \in \mathcal{E}_i$ and $x, y \in H$ be such that $x \in f_i(F)$ and $y \in F \backslash f_i(F)$. Then $(x, y) \in R^*$ and $(y, x) \notin R^*$ (where R^* is given by (10)).*

Proof. First we show that $(x, y) \in R^*$. If $F = H$ then $x \in f_i(H)$. Let $\emptyset$ denote the empty history (recall $\emptyset$ is a prefix of every history). Then $(x, \emptyset) \in R_3$ and $(\emptyset, y) \in R_5$. Thus $(x, y) \in R^*$. Consider now the case where $F \neq H$. Then (since $x, y \in F$) there exist $x_0, y_0 \in D_i$ such that $F = \overrightarrow{I_i}(x_0)$, $y_0 \in I_i(x_0)$, x_0 is a prefix of x and y_0 is a prefix of y. Since $x \in f_i(F)$,

$$(x, x_0) \in R_4 \tag{13}$$

[19] Proof: since $(ha, h) \in R_2$, $\exists h_0 \in I_i(h)$ such that $h_0, h_0 a \in f_i(E)$ where where $E = \overrightarrow{I_i}(h_0)$. Hence, by definition of R_2, $(h'a, h') \in R_2$, for every $h' \in I_i(h_0) = I_i(h)$.

and, by Assumption A1, $x_0 \in f_i(F)$. Thus

$$(x_0, y_0) \in R_1. \tag{14}$$

Hence, since $(y_0, y) \in R_5$, it follows from (13) and (14) that $(x, y) \in R^*$.

Next we show that $(y, x) \notin R^*$. We will show that if $\langle x_1, ..., x_m \rangle$ $(m \geq 2)$ is a sequence in H with $x_1 \in F \backslash f_i(F)$ for some $F \in \mathcal{E}_i$, and, for all $j = 1, ..., m-1$, $(x_j, x_{j+1}) \in R$ (where R is defined in (9)) then $x_m \notin f_i(F)$. For this purpose it will be sufficient to prove the following: $\forall F \in \mathcal{E}_i$, $\forall h_1, h_2 \in H$

$$\text{if } h_1 \in F \backslash f_i(F) \text{ and } (h_1, h_2) \in R \text{ then } h_2 \in F \backslash f_i(F). \tag{15}$$

Let $F \in \mathcal{E}_i$, $h_1 \in F \backslash f_i(F)$ and $(h_1, h_2) \in R$. We need to consider all the possible cases.

Suppose that $(h_1, h_2) \in R_1$. Then $h_1 \in D_i$, $h_2 \in I_i(h_1)$ and $h_1 \in f_i(E)$ where $E = \overrightarrow{I_i}(h_1)$. Since $h_1 \in F$, by Lemma 1 $F \supseteq E$. Thus, since $h_2 \in E$, $h_2 \in F$. Suppose that $h_2 \in f_i(F)$. Then $h_2 \in f_i(F) \cap E$ and thus, by Arrow's Axiom, $f_i(E) = f_i(F) \cap E$, so that $h_1 \in f_i(F)$, contradicting our hypothesis. Hence $h_2 \in F \backslash f_i(F)$.

Suppose that $(h_1, h_2) \in R_2$. Then $h_2 \in D_i$ and $h_1 = h_2 a$ for some $a \in A(h_2)$ and

$$\exists h \in I_i(h_2) \text{ such that } h, ha \in f_i(E) \text{ where } E = \overrightarrow{I_i}(h_2). \tag{16}$$

Let $x \in H$ be the prefix of $h_2 a$ such that $F = \overrightarrow{I_i}(x)$. By Condition C (since $h_2 \in D_i$), $h_2 a \notin D_i$ and thus x is a prefix of h_2, so that $F \supseteq E$.[20] Thus $h_2 \in F$. If $h_2 \in f_i(F)$ then $f_i(F) \cap E \neq \varnothing$ and thus, by Arrow's Axiom, $f_i(E) = f_i(F) \cap E$, so that $h_2 \in f_i(E)$. It follows from this, (16) and Assumption A3 that $h_2 a \in f_i(E)$ and thus $h_2 a \in f_i(F)$, contradicting the hypothesis that $h_2 a = h_1 \in F \backslash f_i(F)$. Hence $h_2 \in F \backslash f_i(F)$.

Next we show that $(h_1, h_2) \notin R_3$. If $(h_1, h_3) \in R_3$ then $h_1 \in f_i(H)$ and thus (since $h_1 \in F$) $f_i(H) \cap F \neq \varnothing$ and by Arrow's Axiom $f_i(F) = f_i(H) \cap F$ so that $h_1 \in f_i(F)$, contradicting the hypothesis that $h_1 \in F \backslash f_i(F)$.

Suppose that $(h_1, h_2) \in R_4$. Then $h_2 \in D_i$, h_2 is a prefix of h_1 and $h_1 \in f_i(E)$ where $E = \overrightarrow{I_i}(h_2)$. Since $h_1 \in E \cap F$, $E \cap F \neq \varnothing$ and thus (see Footnote 4) either $E \subseteq F$ or $F \subseteq E$. It cannot be that $F \subseteq E$ because in this case (since $h_1 \in f_i(E) \cap F$) by Arrow's Axiom $f_i(F) = f_i(E) \cap F$ and thus $h_1 \in f_i(F)$, contradicting our hypothesis. Hence it must be $E \subseteq F$ so that, since $h_2 \in E$, $h_2 \in F$. Suppose that $h_2 \in f_i(F)$. Then $h_2 \in f_i(F) \cap E$ and thus, by Arrow's Axiom, $f_i(E) = f_i(F) \cap E$; hence, since $h_1 \in f_i(E)$, $h_1 \in f_i(F)$, contradicting our hypothesis. Hence $h_2 \in F \backslash f_i(F)$.

Suppose that $(h_1, h_2) \in R_5$. Then h_1 is a prefix of h_2 and thus, since $h_1 \in F$, $h_2 \in F$. If $h_2 \in f_i(F)$ then, by Assumption A1, $h_1 \in f_i(F)$, contradicting our hypothesis. Hence $h_2 \in F \backslash f_i(F)$. $\square$

Proof of Proposition 1. $(a) \Rightarrow (b)$ Let $\precsim_i$ be a total pre-order that rationalizes $\langle H, \mathcal{E}_i, f_i \rangle$ and satisfies Properties PL1 and PL2i. By Proposition 4, $\langle H, \mathcal{E}_i, f_i \rangle$ satisfies Arrow's Axiom and Assumption A1. We need to show that Assumptions A2 and A3 are also satisfied. Let $h \in D_i$ and $F \in \mathcal{E}_i$ and suppose that $h \in f_i(F)$. We want to show that $ha \in f_i(F)$ for some $a \in A(h)$. Let $E = \overrightarrow{I_i}(h)$. By Lemma 1, $F \supseteq E$. Thus, by Arrow's Axiom (since $h \in f_i(F) \cap E$), $f_i(E) = f_i(F) \cap E$. Hence $h \in f_i(E)$ and it will be enough to show that $ha \in f_i(E)$ for some $a \in A(h)$. Since $h \in f_i(E)$ and, by hypothesis, $f_i(E) = \{x \in E : x \precsim_i y, \forall y \in E\}$,

$$h \precsim_i y, \forall y \in E. \tag{17}$$

By (1) of Property PL2i there exists an $a \in A(h)$ such that $ha \precsim_i h$. Thus, by (17) and transitivity of $\precsim_i$, $ha \precsim_i y, \forall y \in E$ and thus $ha \in f_i(E)$. Thus Assumption A2 holds. To prove that Assumption A3 is satisfied, let $h \in D_i$, $a \in A(h)$ and $F \in \mathcal{E}_i$ be such that $h, ha \in f_i(F)$. Fix an arbitrary $h' \in I_i(h) \cap f_i(F)$. We need to show that $h'a \in f_i(F)$. Letting $E = \overrightarrow{I_i}(h)$, by the same argument used above we have that $f_i(E) = f_i(F) \cap E$, so that $h, ha \in f_i(E)$ and $h' \in I_i(h) \cap f_i(E)$ and it is thus sufficient to show that $h'a \in f_i(E)$. Since $ha \in f_i(E)$ and, by hypothesis, $f_i(E) = \{x \in E : x \precsim_i y, \forall y \in E\}$, $ha \precsim_i h$. Thus, by (2) of Property PL2i, $h'a \precsim_i h'$. Since $h' \in f_i(E)$, $h' \precsim_i y, \forall y \in E$. Thus, by transitivity of $\precsim_i$, $h'a \precsim_i y, \forall y \in E$ and therefore $h'a \in f_i(E)$.

$(b) \Rightarrow (a)$ Let $\langle H, \mathcal{E}_i, f_i \rangle$ be a choice frame of player i that satisfies Arrow's Axiom and Assumptions A1-A3. Let R^* be the relation defined in (10). Then R^* is transitive as well as reflexive (because R_5 is reflexive - see Remark 2 - and $R_5 \subseteq R^*$). Let $\precsim_i$ be a total pre-order that extends R^* (see Definition 6 and Proposition 3). Since $R_5 \subseteq R^* \subseteq \precsim_i$, $\precsim_i$ satisfies Property PL1. Next we show that $\precsim_i$ satisfies Property PL2i. Fix an arbitrary $h \in D_i$ and let $E = \overrightarrow{I_i}(h) \in \mathcal{E}_i$. By definition of choice frame, $f_i(E) \neq \varnothing$. Fix an arbitrary $x_0 \in f_i(E)$ and let $h_0 \in I_i(h)$ be the prefix of x_0 in $I_i(h)$. Then, by Assumption A1, $h_0 \in f_i(E)$. Thus, by Assumption A2, there exists an $a \in A(h_0) = A(h)$ such that $h_0 a \in f_i(E)$. Hence, by (5), $(ha, h) \in R_2$ and therefore (since R_2 is a subset of $\precsim_i$) $ha \precsim_i h$. Thus we have proved part (1) of Property PL2i. To prove part (2) of Property PL2i, fix an arbitrary $h \in D_i$ and an arbitrary $a \in A(h)$ and suppose that $ha \precsim_i h$. We have to show that $h'a \precsim_i h'$ for all $h' \in I_i(h)$. Since $(h, ha) \in R_5 \subseteq R^*$ if $(ha, h) \notin R^*$ then, by definition of extension (see Definition 6) $ha \not\precsim_i h$, contradicting our supposition. Thus $(ha, h) \in R^*$. Hence, by Corollary 1, $(ha, h) \in R_2$ and thus (see Remark 3) $(h'a, h') \in R_2$, for all $h' \in I_i(h)$. Since $R_2 \subseteq R^* \subseteq \precsim_i$, $h'a \precsim_i h'$ for all $h' \in I_i(h)$.

It remains to show that $\precsim_i$ rationalizes $\langle H, \mathcal{E}_i, f_i \rangle$. Fix an arbitrary $E \in \mathcal{E}_i$, $h \in f_i(E)$ and $h' \in E$. Then $(h, h') \in R^*$.[21] Thus $f_i(E) \subseteq \{h \in E : hR^*h', \forall h' \in E\}$ so that, since R^* is a subset of $\precsim$, $f_i(E) \subseteq \{h \in E : h \precsim h', \forall h' \in E\}$. Conversely, let $h \in E$ be such that $h \precsim h', \forall h' \in E$. We need to show that $h \in f_i(E)$. Fix an arbitrary $h_0 \in f_i(E)$. Suppose that $h \notin f_i(E)$. Then, by Lemma 3, $(h_0, h) \in R^*$ and $(h, h_0) \notin R^*$. Thus, since $\precsim_i$ is an extension of R^* (see Definition 6), $(h, h_0) \notin \precsim_i$, contradicting our hypothesis

[20] Without Condition C it is possible that $h_2 a \in D_i$ and that $F = \overrightarrow{I_i}(h_2 a)$, in which case $h_2 \notin F$.

[21] The argument is the same as in the first part of the proof of Lemma 3: if $E = H$ then $(h, \emptyset) \in R_3$ and $(\emptyset, h') \in R_5$; if $E \neq H$ then, $(h, x_0) \in R_4$, $(x_0, y_0) \in R_1$ and $(y_0, h') \in R_5$, where $x_0, y_0 \in D_i$ are such that $E = \overrightarrow{I_i}(x_0)$, $y_0 \in I_i(x_0)$, x_0 is a prefix of h and y_0 is a prefix of h'.

that $h \precsim h', \forall h' \in E$.

Proof of Proposition 2. Let $\precsim \ \in \bigcap_{i\in N} \mathcal{P}_i$. By Proposition 1, for every $i \in N$, every element of $\mathcal{P}_i$ satisfies Property PL1. Thus $\precsim$ satisfies PL1. Now fix an arbitrary decision history h and let i be the player to whom it belongs. By Property PL2i of Proposition 1, $\exists a \in A(h)$ such that $ha \precsim h$ and, $\forall a \in A(h)$, if $ha \precsim h$ then $h'a \precsim h', \forall h' \in I_i(h)$. Thus $\precsim$ satisfies also Property PL2. Hence $\precsim$ is a plausibility order.

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