Judgment Aggregation Rules Based on Minimization

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ABSTRACT

Many voting rules are based on some minimization principle. Likewise, in the field of logic-based knowledge representation and reasoning, many belief change or inconsistency handling operators also make use of minimization. Surprisingly, minimization has not played a major role in the field of judgment aggregation, in spite of its proximity to voting theory and logic-based knowledge representation and reasoning. Here we make a step in this direction and study six judgment aggregation rules; two of them, based on distances, have been previously defined; the other four are new, and all inspired both by voting theory and knowledge representation and reasoning. We study the inclusion relationships between these rules and address some of their social choice theoretic properties.

Categories and Subject Descriptors

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General Terms

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Judgment aggregation, voting theory, aggregation rules, distance-based merging

1. INTRODUCTION

In voting theory and in computational social choice, a large body of work focuses on specific voting rules: how their winner sets compare to each other; their social choice-theoretic properties; the computational and communication complexity of winner determination; the theoretical and experimental study of manipulability and control; the amount of information necessary to determine the outcome; *etc.*

A judgment aggregation problem is specified by a set of logically related issues, an agenda, on which the agents cast judgments. The judgments are typically boolean evaluations of the agenda issues. A judgment aggregation rule amalgamates the individual judgments into a collective set of judgments, which should adhere to the logical relations of the agenda issues. Unlike in voting, the bulk of the research in judgment aggregation focuses on possibility and impossibility results: typically, one looks for minimal conditions on the structure of the agenda, on the allowed judgment sets, or on the properties of the logical system, implying the existence or the non-existence of judgment aggregation rules satisfying a small set of desirable properties (such as nondictatorship, unanimity, independence *etc.*); or else, one looks for a characterization of all judgment aggregation rules satisfying a set of properties, possibly under some domain restrictions. But the focus on specific rules and their properties has been the topic of few papers. Still, there are a few exceptions, that we list now.

- the *premise-based* procedure has been introduced in [13] under the name "issue-by-issue voting" and studied in [7, 19]. For this procedure, the agenda is assumed to be partitioned into two subsets: *premises* and *conclusions*. The premises are logically independent. The individuals vote the premises and the majority on each premise is used to find the collective outcome for that premise. From these collective outcomes on the premises, the collective conclusions are derived using either the logical relationships among, or some external constraints regarding, the agenda issues. On the other hand, in the conclusion-based procedure individuals decide privately on the premises and express publicly only their judgments on the conclusions.

- The more general *sequential* procedures [15, 4, 14] proceed this way: the elements of the agenda are considered sequentially, following a fixed linear order over the agenda (corresponding for instance to temporal precedence or to priority) and earlier decisions constrain later ones. Collective consistency is guaranteed by definition. Of course, in the general case, the result depends on the choice of the order, *i.e.* it is (*path-dependent*). Premise-based procedures are specific instances of sequential procedures.

- *Quota-based* rules [4, 2] are a class of rules where each proposition of the agenda is associated with a quota, and the proposition is accepted only if the proportion of individuals accepting it is above the quota. For example, uniform rules take the same quota for all elements of the agenda. The majority rule is a special case of quota-based rules. In [4] sequential quota rules are also considered.

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- *Distance-based* rules [18, 20] assume a predefined distance between judgment sets and/or profiles and choose as collective outcome the consistent judgment sets which are closest (for some notion of closeness) to the individual judgments (see Section 2.4).

Even if a few families of judgment aggregation rules have been proposed and studied, still the focus on the research is more on the search for impossibility theorems and axiomatic characterizations of families of rules, which contrasts with voting theory, where voting rules are defined and studied *per se*.

In voting theory, quite a number of rules are based on some *minimization* (or maximization) process: for instance, *Kemeny*, *Dodgson*, *Slater*, *ranked pairs*, *maximin* etc. (We shall not recall the definition of all these voting rules; the reader can refer, for instance, to [1] for a survey.) Minimization is also a common way of defining reasoning rules (such as belief revision operators, inconsistency handling procedures, or nonmonotonic inference rules) in the community of logic-based knowledge representation and reasoning. Namely, typically one deals with inconsistency by looking for maximal consistent subsets of an inconsistent knowledge base. Belief revision often amounts to incorporating a piece of information to a knowledge base while minimizing the information loss from the initial knowledge base. Similar minimization processes are at work in reasoning about actions, belief update and belief merging.

In contrast, with the exception of distance-based rules, minimization has rarely been considered for judgment aggregation. Here we aim at filling this gap by proposing and studying a few aggregation rules that we argue to be among the most natural minimization-based rules. Our rules maximize the portion of a profile we wish to keep. The way such a maximization is defined depends on the specific rule. Thus, the maximization operated by our aggregation rules is equivalent to minimizing the portion of a profile we wish to remove. In other words, we call our rules "based on minimization", but we could as well say that our rules are based on maximization. Four of the rules we introduce here are new, while two of them correspond, up to some minor details, to judgment aggregation rules already proposed in the literature. We relate them to similar rules in voting theory and/or knowledge representation and reasoning. We study their interrelationships by showing that in most cases, the proposed rules are inclusion-wise incomparable. We also study their links with existing aggregation rules such as sequential or quota-based rules, and some of their social choice theoretic properties (majority-preservation, unanimity, monotonicity, reinforcement). It is important to note that none of the rules introduced in this paper satisfies *independence* (neither do sequential and distance-based rules). For many agendas, the independence condition is sufficient for an impossibility result to occur [16]. Moreover, when the propositions are logically related, independence is a controversial condition [19].

2. JUDGMENT AGGREGATION RULES

2.1 General definitions

Let L be a propositional language built on a finite set of propositional symbols PS. Lastly, if S is a finite set of formulas of L then $\bigwedge(S)$ is the conjunction of all formulas in S. Cn denotes logical closure, that is, $Cn(S) = \{\alpha \in L \mid S \models \alpha\}$.

Definition 1 (agendas, judgment sets, profiles)

- *an* agenda *is a finite set* $X = \{\varphi_1, \neg\varphi_1, \ldots, \varphi_m, \neg\varphi_m\}$ *of propositional formulae of L, consisting of pairs of propositions* $\varphi_i, \neg\varphi_i$*, and containing neither tautologies not contradictions. The* pre-agenda $[X]$ *associated with X is* $[X] = \{\varphi_1, \ldots, \varphi_m\}$.
- *a* judgment set *over X is a subset of X. A judgment set A is* complete *if for every pair* $\{\varphi, \neg\varphi\}$ *in X, A contains either* φ *or* $\neg\varphi$. *A judgment set A is* consistent *if* $\bigwedge\{\varphi_j | \varphi_j \in A\}$ *is satisfiable.*
- *an n-*voter profile *over X is a collection* $P = \langle A_1, \ldots, A_n \rangle$ *where each* A_i *is a consistent and complete*[1] *judgment set.*

We now define judgment aggregation rules.[2] As in voting theory, we distinguish between deterministic rules, mapping a profile to a single collective judgment set, and nondeterministic rules (or correspondences), mapping a profile to a nonempty set of collective judgment sets.

Definition 2 (judgment aggregation rules)

- *a* deterministic judgment aggregation rule *is a mapping* $f_{n,X}$ *associating with every profile* $P = \langle A_1, \ldots, A_n \rangle$ *a consistent judgment set* $f_{n,X}(P)$. *A deterministic aggregation rule* $f_{n,X}$ *is* complete *if for every profile P,* $f_{n,X}(P)$ *is complete.*
- *a* nondeterministic judgment aggregation rule *(or* judgment aggregation correspondence*) is a mapping* $F_{n,X}$ *associating with every profile P a nonempty set of consistent judgment sets* $F_{n,X}(P)$.

Most of the time, when referring to judgment aggregation rules we will keep n and X implicit when they are clear from the context, *i.e.*, $f_{n,X}$ (resp. $F_{n,X}$) will be simply denoted as f (resp. F). Also, by a slight abuse of language, if $P = \langle A_1, \ldots, A_n \rangle$, then we will write $f(A_1, \ldots, A_n)$ and $F(A_1, \ldots, A_n)$ instead of $f(\langle A_1, \ldots, A_n \rangle)$ and $F(\langle A_1, \ldots, A_n \rangle)$.

As in voting theory, a rule can be obtained from a correspondence using a tie-breaking mechanism. In the rest of the paper we focus on nondeterministic rules, unless we state the contrary.[3]

Definition 3 (majoritarian aggregation)
The majority aggregation rule m is defined as: for every profile P, $M(P)$ *is a singleton judgment set* $\{m(P)\}$ *such that for every* $\psi \in X$, $m(P)$ *contains* ψ *if and only if a majority of agents have* ψ *in their judgment set, that is, if and only if* $\#\{i | \psi \in A_i\} > \frac{n}{2}$.
A profile P is majority-consistent *if* $m(P)$ *is a consistent judgment set. A judgment aggregation rule F is* majority-preserving *if, for every majority-consistent profile P,* $F(P) = M(P)$.

Aggregation conditions such as *anonymity* (a permutation on the individual judgment sets does not alter the collective outcome) and

[1] In judgment aggregation consistent and complete judgment sets are usually assumed. However, while consistency seems an indispensable requirement, completeness can be dismissed, at least in some contexts. Some works [10, 8] investigated what happens if we allow voters to abstain from expressing judgments on some propositions in the agenda. We could also define profiles more generally by allowing individual judgment sets to be incomplete. Most of our results would not be altered, but some of the definitions, especially distance-based rules, would be more complicated.

[2] Strictly speaking, a rule is a function; we keep the terminology "rule" for the sake of the parallel with voting theory.

[3] One may also want to require that not only the output of a judgment aggregation rule is a single judgment set A, but that this judgment set is itself *complete*. Doing this amounts at having another tie-breaking rule which, in case of a tie between φ and $\neg\varphi$, specifies how to break it.

neutrality (the elements of the agenda are aggregated in the same way) can be defined as usual (and we omit their formal definitions).

There are two different views of aggregation rules: either we see the output as a mere collection of consistent judgment sets, or we see it as a closed logical theory.

Definition 4

Given a judgment aggregation rule F, and a profile P, we define the logical theory $T_F(P) = \bigcap\{Cn(J) \mid J \in F(P)\}$.

Let F and F' be two aggregation rules. F and F' are equivalent *if for every profile P we have $T_F(P) = T_{F'}(P)$. F is* at least as discriminant *as F' if for every profile P we have $T_F(P) \supseteq T_{F'}(P)$. F and F' are* incomparable *if there exists two profiles P and Q such that $T_F(P) \not\subseteq T_{F'}(P)$ and $T_F(Q) \not\subseteq T_{F'}(Q)$.*

Thus, a formula α is in $T_F(P)$ if and only if it can be inferred from every judgment set in $F(P)$. Note that $T_F(P)$ being the intersection of consistent closed logical theories, it is itself a consistent closed theory.

We now give a series of aggregation rules. In the following we use the abbreviation *maxcard* for *of maximal cardinality*.

2.2 Rules based on maximal consistent judgment sets

The first rule we consider is called the *Young* rule for judgment aggregation, by analogy with the Young rule in voting, which outputs the candidate x minimizing the number of voters to remove from the profile so that x becomes a Condorcet winner.

Definition 5 (Young rule for judgment aggregation)

Given a profile $P = \langle A_1, \ldots, A_n\rangle$ and a subset of agents $J \subseteq \{1, \ldots, n\}$, the restriction of P to J is $P_J = \langle A_j, j \in J\rangle$, and is called a subprofile *of P. Let $MSP(P)$ be the set of maxcard majority-consistent subprofiles of P. Then the* Young *judgment aggregation rule Y maps P to $R_Y(P) = \{m(P_J) \mid P_J \in MSP(P)\}$.*

Intuitively, this rule consists in removing a minimal number of agents so that the profile becomes majority-consistent. Or, equivalently, we maximize the number of voters we keep of a given profile. Obviously, if the profile P is majority-consistent, then no voter needs to be removed and $Y(P) = \{m(P)\}$, hence Y is majority-preserving.

Example 1 *Consider the pre-agenda $[X] = \{a, b, c, \alpha = (a \vee b) \wedge c, \beta = a \wedge b\}$, $n = 9$, and $P = \langle A_1, \ldots, A_9\rangle$:*

	a	b	c	$(a \vee b) \wedge c$	$a \wedge b$
A_1	1	1	1	1	1
A_2	1	1	1	1	1
A_3	1	1	0	0	1
A_4	0	1	1	1	0
A_5	0	1	1	1	0
A_6	1	0	0	0	0
A_7	1	0	0	0	0
A_8	0	0	1	0	0
A_9	0	0	1	0	0

We have $m(P) = \{a, b, c, \neg\alpha, \neg\beta\}$. There are three minimal inconsistent subsets in $m(P)$, namely $\{a, c, \neg\alpha\}$, $\{b, c, \neg\alpha\}$, and $\{a, b, \neg\beta\}$. In order to restore the consistency of $m(P)$, it is sufficient to remove one judgment set, and the possible choices are A_1 or A_2 (leading to accept c, $\neg\alpha$, $\neg\beta$), A_3 (leading to accept c, $\neg\beta$), or A_6 or A_7 (leading to accept b, c, $\neg\beta$). Therefore,

$$R_Y(P) = \{\{c, \neg\alpha, \neg\beta\}, \{c, \neg\beta\}, \{b, c, \neg\beta\}\}$$

Note that $T_{R_Y}(P) = Cn(c \wedge \neg\beta)$.

A refinement of R_Y could consist in keeping only the judgment sets in $R_Y(P)$ of maximal size (here, $\{c, \neg\alpha, \neg\beta\}$ or $\{b, c, \neg\beta\}$). We do not explore this here.

The second proposal consists in looking for a minimal subset or a minimal number of formulas in X to remove such that the profile becomes majority-consistent.

Definition 6 (maximal subagenda rule)

Given a profile $P = \langle A_1, \ldots, A_n\rangle$ on an agenda X, $[X]$ the pre-agenda associated with X, and a sub-preagenda $[Y] \subseteq [X]$, the restriction of P to Y is $P^{\downarrow Y} = \langle A_j \cap Y, 1 \le j \le n\rangle$. Let $MSA(P)$ the set of all maximal sub-preagendas $[Y]$ of $[X]$ (with respect to set inclusion) such that $P^{\downarrow Y}$ is majority-consistent. The maximal subagenda *judgment aggregation rule R_{MSA} maps P to $R_{MSA}(P) = \{m(P^{\downarrow Y}) \mid [Y] \in MSA(P)\}$.*

Example 2 *Take the same profile as in Example 1. The maximal majority-consistent sub-preagendas are $\{a, b, c\}$, $\{a, b, \alpha\}$, $\{a, c, \beta\}$, $\{b, c, \beta\}$ and $\{c, \alpha, \beta\}$. Therefore $R_{MSA}(P) = \{\{a, b, c\}, \{a, b, \neg\alpha\}, \{a, c, \neg\beta\}, \{b, c, \neg\beta\}, \{c, \neg\alpha, \neg\beta\}\}$.*

Instead of looking for maximal majority-consistent subagendas with respect to inclusion we may look instead for maxcard majority-consistent subagendas, which leads to the following judgment aggregation rule, which corresponds, up to some minor details and for a specific choice of a distance function, to the *endpoint* judgment aggregation rule defined in [18].

Definition 7 (maxcard subagenda rule)

Let $MCSA(P)$ the set of all maxcard sub-preagendas $[Y]$ of $[X]$ such that $P^{\downarrow Y}$ is majority-consistent. The maxcard subagenda *judgment aggregation rule R_{MCSA} maps P to $R_{MCSA}(P) = \{m(P^{\downarrow Y}) \mid [Y] \in MCSA(P)\}$.*

Example 3 *Take the same profile as in Example 1. The maxcard majority-consistent sub-preagendas are the same $\{a, b, c\}$, $\{a, b, \alpha\}$, $\{a, c, \beta\}$, $\{b, c, \beta\}$ and $\{c, \alpha, \beta\}$ as in Example 2. Therefore $R_{MCSA}(P) = \{\{a, b, c\}, \{a, b, \neg\alpha\}, \{a, c, \neg\beta\}, \{b, c, \neg\beta\}, \{c, \neg\alpha, \neg\beta\}\}$.*

R_Y considers a judgment set as a unit, which is either selected or removed as a whole. Similarly, R_{MSA} and R_{MCSA} consider the judgments on the agenda subset as a unit that is to be kept in its entirety or got ridden of. A finer way of defining a judgment rule consists in looking for maximal or maxcard majority-consistent subsets of the set of *elementary pieces of information* consisting each of a pair (element of the agenda, judgment on it elicited from an agent). Equivalently, this comes down to weigh each element of the agenda by the number of agents who support it, and then to look for maxweight subagendas.

Definition 8 (maxweight subagenda rule)

For each $\psi \in X$ and each profile P, let $N(P, \psi) = \#\{i, \psi \in A_i\}$. For any subagenda $Y \subseteq X$, the weight of Y with respect to P is defined by $w_P(Y) = \sum_{\psi \in Y} N(P, \psi)$. Let $MWA(P)$ be the set of all consistent subagendas Y of X maximizing w_P. The maxweight subagenda *judgment aggregation rule R_{MWA} maps P to $R_{MWA}(P) = \{Y \mid Y \in MWA(P)\}$.*

Example 4 *Take again the same profile as in Example 1. We have $N(P, a) = 5$, $N(P, \neg a) = 4$, $N(P, b) = 5$, $N(P, \neg b) = 4$, $N(P, c) = 6$, $N(P, \neg c) = 3$, $N(P, \alpha) = 4$, $N(P, \neg\alpha) = 5$, $N(P, \beta) = 3$, and $N(P, \neg\beta) = 6$. Therefore we have $R_{MWA}(P) = \{\{a, \neg b, c, \alpha, \neg\beta\}, \{\neg a, b, c, \alpha, \neg\beta\}, \{\neg a, \neg b, c, \neg\alpha, \neg\beta\}\}$.*

Although it looks entirely new, we will show soon that this natural rule corresponds to a rule already defined, in a totally different way, in [9].

As it can be observed, R_{MSA}, R_{MCSA} and R_{MWA} are majority-preserving (the proof is straightforward for the first two ones and easy for the third one).

2.3 A rule inspired from the "ranked pairs" voting rule

The following rule is inspired from the ranked pairs rules in voting theory [22]. It consists in fixing first the truth value for the elements of the agenda with the largest majority, and iterate, considering the elements of the agenda in the decreasing order of the number of agents who support them, and fix their value to the majoritarian value as long as this is possible without producing an inconsistency.

Definition 9 (ranked agenda) *Let $Y = \{\varphi \in X | N(P,\varphi) > \frac{n}{2}\}$, and let $\geq_P$ the complete weak order relation on Y defined by $\varphi \succeq_P \psi$ if $N(P,\varphi) \geq N(P,\psi)$. $R_{RA}(P)$ is defined as follows: $A \in R_{RA}(P)$ if there exists a linear order $\succ$ on X refining $\geq$ such that $RA(\succ,P) = A$, where $RA(\succ,P)$ is defined inductively by*

- *reorder the elements of Y following $\succ$, i.e., such that $\varphi_{\sigma(1)} \succ \cdots \varphi_{\sigma(m)}$;*
- *$D := \emptyset$;*
- *for $k := 1$ to m do: if $D \cup \{\varphi_{\sigma(i)}\}$ is consistent then $D := D \cup \{\varphi_{\sigma(i)}\}$;*
- *$RA(\succ,P) := D$.*

Example 5 *Take again the same profile as in Example 1. We have $Y = \{a,b,c,\neg\alpha,\neg\beta\}$, and $c \sim_P \neg\beta >_P a \sim_P b \sim_P \neg\alpha$ (where $\sim_P$ and $>_P$ are respectively the indifference and the strict preference relations induced from $\succeq_P$). Therefore, we have to consider twelve linear orders $\succ_1$: $c \succ \neg\beta \succ a \succ b \succ \neg\alpha$; $\succ_2$: $\neg\beta \succ c \succ a \succ b \succ \neg\alpha$; $\succ_3$: $c \succ \neg\beta \succ a \succ \neg\alpha \succ b$; $\succ_4$: $\neg\beta \succ c \succ a \succ \neg\alpha \succ b$;etc. For each one of these twelve linear orders, c and $\neg\beta$ are considered first (in any order). Since $c \wedge \neg\beta$ is consistent, at this point of the construction we have $D = \{c,\neg\beta\} = \{c,\neg a \vee \neg b\}$. If a is considered next, then, since $D \cup \{a\}$ is consistent, D is updated to $\{a,c,\neg a \vee \neg b\}$. Note that $\bigwedge D \equiv a \wedge \neg b \wedge c$; since D has only one model, there is no need to go further. If b is considered next (i.e., after c and $\neg\beta$) then D is updated to $\{b,c,\neg a \vee \neg b\}$, which has only one model $\neg abc$. Lastly, if $\neg\alpha$ is considered next (i.e., after c and $\neg\beta$) then D is updated to $\{\neg\alpha,c,\neg a \vee \neg b\}$, which is equivalent to $\neg\alpha \wedge c \wedge \neg a \wedge \neg b$. Therefore, $R_{RA}(P) = \{\{a,\neg b,c,\alpha,\neg\beta\},\{\neg a,b,c,\alpha,\neg\beta\},\{\neg a,\neg b,c,\neg\alpha,\neg\beta\}\}$.*

2.4 Distance-based rules

Distance-based judgment aggregation rules [20, 18] are derived from distance-based merging operators for belief bases [12, 11]. The definition we give here is the same as in [9] (who call it "syntactic distance-based belief merging") and is slightly different (and more widely applicable) than the definition in [20, 18].

Let Φ_X be the set of all complete and consistent judgment sets formed from X, that is, the set of all consistent judgment sets containing either φ or $\neg\varphi$ for each $\varphi \in [X]$. Let $d : \Phi_X \times \Phi_X \mapsto \mathbb{R}^+$ be a distance function between judgment sets from Φ_X.[4] Let $\odot : (\mathbb{R}^+)^n \mapsto \mathbb{R}^+$ be a symmetric, non-decreasing aggregation function such that, for every $x, y, x_1, \ldots, x_n \in \mathbb{R}$, has the following properties: $\odot(x,\ldots,x) = x$; $\odot(x_1,\ldots,x_n) = 0$ if and only if $x_1 = \ldots = x_n = 0$. The distance-based judgment aggregation rule $R^{d,\odot}$ induced by d and $\odot$ is defined by:

$$R^{d,\odot}(A_1,\ldots,A_n) = \underset{A \in \Phi_X}{\arg\min}\ \odot(d(A,A_1),\ldots d(A,A_n)).$$

Here we consider only $\odot = \sum$ and $\odot = \max$, and the Hamming distance d_H on complete judgment sets, defined as $d_H(A,A') = |A \setminus A'| + |A' \setminus A|$. [5] Therefore we consider the two rules $R^{d_H,\Sigma}$ and $R^{d_H,max}$, which reduce to only one after the following easy result is established:

Proposition 1 *$R^{d_H,\Sigma}$ and R_{MWA} are equivalent.*

PROOF. Given two complete judgment sets A and A', and $\varphi \in X$, define $h(\varphi,A,A') = 1$ if $\varphi \in (A \setminus A') \cup (A' \setminus A)$ and $h(\varphi,A,A') = 0$ otherwise.

Now, for any profile $P = \langle A_1,\ldots,A_n\rangle$ and any complete judgment set A, we have

$$\begin{aligned}
& \textstyle\sum_{i=1}^n d_H(A,A_i) \\
= & \textstyle\sum_{i=1}^n \sum_{\varphi \in X} h(\varphi,A,A_i) \\
= & \textstyle\sum_{i=1}^n \left(\sum_{\varphi \in A} h(\varphi,A,A_i) + \sum_{\varphi \notin A} h(\varphi,A,A_i)\right) \\
= & \textstyle\sum_{i=1}^n \left(\sum_{\varphi \in A} h(\varphi,A,A_i) + \sum_{\neg\varphi \in A} h(\varphi,A,A_i)\right) \\
= & \textstyle\sum_{i=1}^n \left(\sum_{\varphi \in A} h(\varphi,A,A_i) + \sum_{\varphi \in A} h(\neg\varphi,A,A_i)\right) \\
= & \textstyle\sum_{\varphi \in A} \left(\sum_{i=1}^n h(\varphi,A,A_i) + \sum_{i=1}^n h(\neg\varphi,A,A_i)\right) \\
= & \textstyle\sum_{\varphi \in A} (n - N(P,\varphi) + N(P,\neg\varphi)) \\
= & \textstyle\sum_{\varphi \in A} 2(n - N(P,\varphi)) \\
= & 2n * |A| - 2w_P(A)
\end{aligned}$$

Therefore, $\sum_{i=1}^n d_H(A,A_i)$ is minimum if and only if $A \in MWA(P)$, that is, $w_P(A)$ is maximum. Since every element of $MWA(P)$ is a complete judgment set, $MWA(P)$ is equal to the set of all complete judgment sets minimizing $\sum_{i=1}^n d_H(A,A_i)$, which allows us to conclude that $R^{d_H,\Sigma}$ and R_{WMA} are equivalent. ■

As a consequence, $R^{d_D,\Sigma}$ is majority-preserving. This is however not the case for $R^{d_D,\Sigma}$, which is the only one of our rules failing to satisfy majority-preservation.

Proposition 2 *$R^{d_H,max}$ is not majority-preserving.*

PROOF. Consider the agenda $X = \{a,\neg a,b,\neg b\}$ and $P = \langle\{a,b\},\{a,b\},\{\neg a,\neg b\}\rangle$. Then $R^{d_H,max}(P) = \{\{a,\neg b\},\{\neg a,b\}\}$; however, P is majority-consistent and $M(P) = \{\{a,b\}\}$. ■

3. (NON)INCLUSION RELATIONSHIPS BETWEEN THE RULES

Proposition 3 *We have the following diagram (Table 1), where inc means "inclusion-wise incomparable".*

PROOF. 1. *For all profiles P, $T_{R_{MSA}}(P) \subseteq T_{R_{MCSA}}(P)$.*
If $Y \subset [X]$ is a maxcard consistent sub-preagenda (w.r.t. P) of $[X]$ then it is also a maximal consistent sub-preagenda (w.r.t. P). Now, if $\alpha \in T_{R_{MSA}}(P)$, then α is inferred in every maximal consistent sub-preagenda, and *a fortiori* in every maxcard consistent sub-preagenda, therefore $\alpha \in T_{R_{MCSA}}(P)$.

[4] We recall that d is a distance function if and only if for all $A, A', A'' \in \Phi_X$ we have (i) $d(A,A') = 0$ if and only if $A = A'$, (ii) $d(A,A') = d(A',A)$, and (iii) $d(A,A') + d(A',A'') \geq d(A,A'')$.

[5] We could also consider the "drastic distance" d_D, defined as $d_D(A,A') = 0$ if and only if $A = A'$ and $d_D(A,A') = 1$ otherwise. Taking $d = d_D$ and $\odot = \Sigma$ leads to a judgment aggregation rule that selects the judgment sets given by the highest number of agents, while taking $\odot = \max$ also leads to a rule of no interest.

	R_{MSA}	R_{MCSA}	R_{MWA}	R_{RA}	$R^{d_H,\max}$
R_Y	*inc*	*inc*	*inc*	*inc*	*inc*
R_{MSA}		$\subseteq$	$\subseteq$	$\subseteq$	*inc*
R_{MCSA}			*inc*	*inc*	*inc*
R_{MWA}				*inc*	*inc*
R_{RA}					*inc*

Table 1: Inclusion relations between the judgment aggregation rules.

2. *For all profiles P,* $T_{R_{MSA}}(P) \subseteq T_{R_{MWA}}(P)$.
   If $Y \subset [X]$ is a consistent sub-preagenda maximizing $w_P(Y)$, then $m(P^{\downarrow Y})$ is a maximal consistent subagenda (w.r.t. P). Now, if $\alpha \in T_{R_{MSA}}(P)$, then α is inferred in every maximal consistent sub-preagenda, and *a fortiori* in every maxweight consistent subagenda, therefore $\alpha \in T_{R_{MWA}}(P)$.

3. *For all profiles P,* $T_{R_{MSA}}(P) \subseteq T_{R_{RA}}(P)$. In the construction of $R_{RA}(P)$, let Z be the subset of X composed of the ψ_k such that $\delta \wedge \psi_k$ is consistent. Z is a maximal consistent subagenda w.r.t. P (it is consistent by construction, and maximal because every time a formula ψ_k is rejected, it is because it produces an inconsistency with the formulas already present in δ). Now, if $\alpha \in T_{R_{MSA}}(P)$, then α is inferred in every maximal consistent subagenda, and *a fortiori* in Z, therefore $\alpha \in T_{R_{RA}}(P)$.

4. $R^{d_H,\max}$ *is incomparable with all of the five other rules.*
   Let R be a majority-preserving rule. Take the profile P as in the proof of Proposition 2. Then $a \leftrightarrow \neg b \in T_{R^{d_H,\max}}(P)$, whereas $a \leftrightarrow \neg b \notin T_R(P)$ (since $a \leftrightarrow b \in T_R(P)$); and $a \in T_R(P)$, whereas $a \notin T_{R^{d_H,\max}}(P)$. Therefore, $R^{d_H,\max}$ is incomparable with all of the five other rules.

5. R_Y *is incomparable with* R_{MSA} *and* R_{MCSA}.
   Consider the following profile P, with pre-agenda $[X] = \{a, a \to (b \vee c), b, c, a \to (d \vee e), d, e\}$, and three agents with the following information sets:

a	$a \to (b \vee c)$	b	c	$a \to (d \vee e)$	d	e
+	+	+	−	+	+	−
+	+	−	+	+	−	+
+	−	−	−	−	−	−

   The majoritarian aggregation obtained from this profile is $B = \{a, a \to (b \vee c), \neg b, \neg c, a \to (d \vee e), \neg d, \neg e\}$. The minimal inconsistent subsets of B are $\{a, a \to (b \vee c), b, c\}$ and $\{a, a \to (d \vee e), d, e\}$, therefore, there B has 10 maximal consistent subsets: 9 containing a, two of the three formulas $\{a \to (b \vee c), \neg b, \neg c\}$ and two of the three formulas $\{a \to (d \vee e), \neg d, \neg e\}$, and one equal to $B \setminus \{a\}$. These 10 maximal consistent subsets correspond to 10 maximal subagendas; the only maxcard consistent subagenda is $B \setminus \{a\}$, and in this subagenda of B, $\neg a$ is inferred. Therefore, $T_{R_{MCSA}}(P) \models \neg a$. Now, all sub-profiles of P of size two is majority-consistent, and each of them accepts a, therefore $T_{R_Y}(P) \models a$. Therefore, R_Y and R_{MCSA} are incomparable. For $T_{R_Y}(P) \not\subseteq T_{R_{MSA}}(P)$, take the same profile as above and note that $a \in T_{R_Y}(P)$ but $a \notin T_{R_{MSA}}(P)$. For $T_{R_{MSA}}(P) \not\subseteq T_{R_Y}(P)$, assume the pre-agenda is extended with another agenda item f, on which the agents vote +, +, -. We have $f \in T_{R_{MSA}}$ but $f \notin T_{R_Y}$.

6. $R_{MWA}(P)$ *is incomparable with* R_{MCSA}.
   Take the following seven agent profile P:

	a	b	$a \wedge b$
3×	+	+	+
2×	+	−	−
2×	−	+	−

   We obtain that $R_{MWA}(P) = \{\{a, b, a \wedge b\}\}$, while $R_{MCSA}(P) = \{\{a, b\}, \{a, \neg a \vee \neg b\}, \{b, \neg a \vee \neg b\}\}$. Thus $a \in T_{R_{MWA}}(P)$ whereas $a \notin T_{R_{MCSA}}(P)$. For the converse, in the example of item 5, we have $\neg a \notin T_{R_{MWA}}(P)$ and $\neg a \in T_{R_{MCSA}}(P)$.

7. $R_{RA}(P)$ *is incomparable with* R_{MCSA}.
   Same profile P as in point 5 of this proof. We have that $T_{R_{RA}}(P) \models a$. Hence $a \in T_{R_{RA}}(P)$ whereas $\neg a \in R_{MCSA}(P)$, see item 5.

8. $R_{MWA}(P)$ *is incomparable with* R_Y.
   Consider the pre-agenda, introduced in [21]:
   $[X] = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8, \varphi_9, \varphi_{10}, \varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{14}\}$. The set of admissible judgment sets Φ_X is:
   $A^1 = \{\varphi_1, \neg\varphi_2, \neg\varphi_3, \varphi_4, \neg\varphi_5, \neg\varphi_6, \varphi_7, \neg\varphi_8, \neg\varphi_9, \varphi_{10}, \neg\varphi_{11}, \neg\varphi_{12}, \varphi_{13}, \varphi_{14}\}$
   $A^2 = \{\neg\varphi_1, \varphi_2, \neg\varphi_3, \neg\varphi_4, \varphi_5, \neg\varphi_6, \neg\varphi_7, \varphi_8, \neg\varphi_9, \neg\varphi_{10}, \varphi_{11}, \neg\varphi_{12}, \varphi_{13}, \varphi_{14}\}$
   $A^3 = \{\neg\varphi_1, \neg\varphi_2, \varphi_3, \neg\varphi_4, \neg\varphi_5, \varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \neg\varphi_{10}, \neg\varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{14}\}$
   $A^4 = \{\neg\varphi_1, \neg\varphi_2, \neg\varphi_3, \neg\varphi_4, \neg\varphi_5, \neg\varphi_6, \neg\varphi_7, \neg\varphi_8, \neg\varphi_9, \neg\varphi_{10}, \neg\varphi_{11}, \neg\varphi_{12}, \neg\varphi_{13}, \varphi_{14}\}$
   $A^5 = \{\neg\varphi_1, \neg\varphi_2, \neg\varphi_3, \neg\varphi_4, \neg\varphi_5, \varphi_6, \neg\varphi_7, \varphi_8, \neg\varphi_9, \neg\varphi_{10}, \neg\varphi_{11}, \neg\varphi_{12}, \neg\varphi_{13}, \neg\varphi_{14}\}$
   Let the profile be $P = (A^1, A^2, A^3)$.
   We have that $R_Y(P) = \{A^1, A^2, A^3\}$, hence $\varphi_{13} \in R_Y(P)$. However, as it can be observed in Table 8, the agents are unanimous on issue φ_{13}, but the only judgment set selected by $R^{d,\Sigma}(A_1, A_2, A_3)$ is A^4 and $\neg\varphi_{13} \in A^4$.

$A \in \Phi_X$	$d_H(A, A_1)$	$d_H(A, A_2)$	$d_H(A, A_3)$	Σ
A^1	0	8	8	16
A^2	8	0	8	16
A^3	8	8	0	16
A^4	5	5	5	15
A^5	8	6	4	18

Table 2: The sum of Hamming distances from an element in the set Φ_X to each of the agent's judgment sets. The judgment set chosen by $R_{MWA}(P)$ is A^4.

9. $R_{RA}(P)$ *is incomparable with* R_Y.
   We do not have $T_{R_{RA}}(P) \subseteq T_{R_Y}(P)$ as a consequence of item 3 and 5. For the converse, consider the following profile P, with pre-agenda $[X] = \{p, q, p \wedge q, r, s, r \wedge s, t\}$ and 18 agents into six different groups:

	p	q	$p \wedge q$	r	s	$r \wedge s$	t
1×	+	+	+	−	+	−	+
3×	+	+	+	−	+	−	−
4×	+	+	+	+	−	−	−
2×	+	−	−	+	−	−	−
4×	+	−	−	+	+	+	+
4×	−	+	−	+	+	+	+

We easily check that the minimal number of agents to remove so as to make the profile majority-consistent is two, and that these agents are the two agents of the fourth group. Therefore, $t \in T_{R_Y}(P)$, whereas $t \notin T_{R_{RA}}(P)$.

10. *$R_{RA}(P)$ is incomparable with R_{MWA}.*
Same profile as in item 8. We have $\phi_{13} \in T_{R_{RA}}(P)$, whereas $\neg\phi_{13} \in T_{R_{MWA}}(P)$.

Gathering all results, we get the diagram.
■

4. SOCIAL CHOICE THEORETIC PROPERTIES

We consider now some important social choice-theoretic properties for judgment aggregation, and identify which of our six rules satisfy them. Note that obviously, all our six rules satisfy neutrality and anonymity.

4.1 Unanimity

Dietrich and List [6] define unanimity as follows.

Definition 10 (Unanimity principle [6]) *For all profiles $\langle A_1, \ldots, A_n\rangle$ in the domain of the aggregation rule f and all $\varphi \in X$, if $\varphi \in A_i$ for all individuals i, then $\varphi \in f(A_1, \ldots, A_n)$.*

Given that we consider nondeterministc rules, we have two versions of unanimity, whether the unanimously approved formula must be in *some* collective judgment set (weak unanimity) or in *all* judgment sets (strong unanimity). Let F be a judgment aggregation rule.

Definition 11

- *F satisfies* weak unanimity *if for every profile $P = \langle A_1, \ldots, A_n\rangle$ and all $\varphi \in X$, if $\varphi \in A_i$ for all i, then there exists a judgment set $A \in F(P)$ such that $\varphi \in A$.*
- *F satisfies* strong unanimity *if for every profile $P = \langle A_1, \ldots, A_n\rangle$ and all $\varphi \in X$, if $\varphi \in A_i$ for all i, then for all judgment sets $A \in F(P)$ we have $\varphi \in A$.*

Clearly, strong unanimity implies weak unanimity.

Proposition 4

- *R_Y satisfies strong (and weak) unanimity;*
- *R_{MSA} satisfies weak unanimity but not strong unanimity;*
- *R_{MCSA} does not satisfy weak (or strong) unanimity.*
- *R_{MWA} does not satisfy weak (or strong) unanimity.*
- *R_{RA} satisfies strong (and weak) unanimity.*
- *$R^{d_H,max}$ does not satisfy weak (or strong) unanimity.*

PROOF. 1. *R_Y satisfies strong (and weak) unanimity*
Straightforward from the fact that if α is unanimously accepted by N, it is consequently unanimously selected by all consistent subsets of N.

2. *R_{MSA} satisfies weak unanimity but not strong unanimity; R_{MCSA} does not satisfy weak (or strong) unanimity*
Let P be a profile on an agenda X, and $\varphi \in X$ such that all agents in P agree on φ. There exists a maximal consistent agenda containing φ, and in this subagenda P entails φ, therefore R_{MSA} satisfies weak unanimity.

Now, consider again the profile P of point 5 of Proposition 3. Because there is a maximal consistent subagenda of P containing $\neg a$, R_{MSA} does not satisfy strong unanimity, and because the only maxcard consistent subagenda of P contains $\neg a$ (and does not contain a), R_{MCSA} does not even satisfy weak unanimity.

3. *R_{MWA} does not satisfy weak (or strong) unanimity*

See again the counterexample can be found in [21], which we presented in point 7 of the proof of Proposition 3.

4. *R_{RA} satisfies strong (and weak) unanimity*
Let P be a profile and $Y_P \subseteq X$ be the subset of the agenda consisting of all elements on which there is unanimity among the agents. Because individual judgment sets are consistent, the conjunction of all elements of Y is consistent. Now, when computing $R_{RA}(P)$, the elements of Y are considered first, and whatever the order in which they are considered, they are included in δ because no inconsistency arises. Therefore, for all $\alpha \in Y_P$ and all $J \in R_{RA}(P)$, we have $\alpha \in J$.

5. *$R^{d_H,max}$ does not satisfy weak (or strong) unanimity*
Consider the pre-agenda
$[X] = \{a, b, c, d, \underbrace{(a \wedge b \wedge c \wedge d) \vee (\neg a \wedge \neg b \wedge \neg c \wedge \neg d)}_{\alpha}\}$,
and the profile $P = \langle A_1, A_2\rangle$ consisting of these two judgment sets: $A_1 = \{a, b, c, d, \alpha\}$ and $A_2 = \{\neg a, \neg b, \neg c, \neg d, \alpha\}$. The elements of Φ_X and their distances to A_1 and A_2 are given in Table 3. As it can be observed from the table, the $R^{d_H,max}(P)$ selects all $A \in \Phi_X$ for which $max(d_H(A, A_1), d_H(A, A_2)) = 3$. For all such A it holds that $\alpha \notin A$. Since $\alpha \in A_1$ and $\alpha \in A_2$, for this P, $R^{d_H,max}$ does not satisfy the weak unanimity property.

$A \in \Phi_X$	$d_H(A, A_1)$	$d_H(A, A_2)$	max
$\{a, b, c, d, \alpha\}$	0	4	4
$\{a, b, c, \neg d, \neg\alpha\}$	2	4	4
$\{a, b, \neg c, d, \neg\alpha\}$	2	4	4
$\{a, b, \neg c, \neg d, \neg\alpha\}$	3	3	3
$\{a, \neg b, c, d, \neg\alpha\}$	2	4	4
$\{a, \neg b, c, \neg d, \neg\alpha\}$	3	3	3
$\{a, \neg b, \neg c, d, \neg\alpha\}$	3	3	3
$\{a, \neg b, \neg c, \neg d, \neg\alpha\}$	4	2	4
$\{\neg a, b, c, d, \neg\alpha\}$	2	4	4
$\{\neg a, b, c, \neg d, \neg\alpha\}$	3	3	3
$\{\neg a, b, \neg c, d, \neg\alpha\}$	3	3	3
$\{\neg a, b, \neg c, \neg d, \neg\alpha\}$	4	2	4
$\{\neg a, \neg b, c, d, \neg\alpha\}$	3	3	3
$\{\neg a, \neg b, c, \neg d, \neg\alpha\}$	4	2	4
$\{\neg a, \neg b, \neg c, d, \neg\alpha\}$	4	2	4
$\{\neg a, \neg b, \neg c, \neg d, \alpha\}$	4	0	4

Table 3: The max of Hamming distances from an element in the set Φ_X to each of the agent's judgment sets.

■

4.2 Monotonicity

There are two monotonicity properties defined in the judgment aggregation literature. The monotonicity on an agenda issue [3, 17] and the monotonicity on a judgment set [5]. Here we consider only monotonicity on an agenda issue, which corresponds to a form of monotonicity in voting.

Definition 12 (Monotonicity [3]) *For all $\varphi \in X$, individual i and pair of i-variant profiles $\langle A_1,\ldots,A_i,\ldots,A_n\rangle$, $\langle A_1,\ldots,A_i^*,\ldots,A_n\rangle \in Dom(F)$, with $\varphi \notin A_i$ and $\varphi \in A_i^*$, [$\varphi \in A$ for some $A \in F(A_1,\ldots,A_n)$ implies $\varphi \in A'$ for some $A' \in F(A_1,\ldots,A_i^*,\ldots,A_n)$]*

Five of our rules satisfy a property related to monotonicity, namely, *insensitivity to reinforcement of approved formulas*. The exception is $R^{d_H,max}$.

Definition 13 *Let P be a profile over X and $\alpha \in X$. P' is called an* α-improvement *of P if $P' = (A_i', A_{-i})$ where (a) $\neg\alpha \in A_i$ and $A_i' = (A_i \setminus \{\neg\alpha\}) \cup \{\alpha\}$, and (b) A_i' is consistent. R satisfies* insensitivity to reinforcement of approved formulas *if for all profiles P such that $\alpha \in T_R(P)$ and all α-improvements P' of P, we have $R(P') = R(P)$.*

Proposition 5 *R_{MSA}, R_{MCSA}, R_{MWA} and R_{RA} satisfy insensitivity to reinforcement of approved formulas. $R^{d_H,max}$ does not satisfy this property.*

PROOF. 1. We consider R_{MSA}. Assume that $\alpha \in T_{R_{MSA}}(P)$. Let $X' \subseteq X$ be a maximal agenda for which $P^{\downarrow X'}$ is majority-consistent. Because $\alpha \in T_{R_{MSA}}(P)$, we must have $\alpha \in X'$. Obviously, $P'^{\downarrow X'}$ is majority-consistent as well and moreover $m(P'^{\downarrow X'}) = m(P^{\downarrow X'})$ (1). Moreover, it entails that all maximal majority-consistent subagenda for P' contain some maximal majority-consistent subagenda for P' (2). Now, let $X' \subseteq X$ be a maximal agenda for which $P'^{\downarrow X'}$ is majority-consistent. If $\alpha \notin m(P'^{\downarrow X'})$ then *a fortiori* $\alpha \notin m(P^{\downarrow X'})$, which contradicts (2). Therefore, $\alpha \in m(P'^{\downarrow X'})$, and because of (2), it is also a maximal majority-consistent subagenda for P. We have shown that the maximal majority-consistent subagendas for P and P' coincide, therefore $R_{MSA}(P) = R_{MCSA}(P')$. The proof for R_{MCSA} is similar.

2. Now, we consider R_{RA}. Let $\alpha \in X$ and assume that $\alpha \in T_{R_{RA}}(P)$. Then all subagendas in $R_{RA}(P)$ contains α. Let P' be an α-improvement of P. Then $N(P',\alpha) > N(P,\alpha)$, $N(P',\neg\alpha) < N(P,\neg\alpha)$, whereas for all $\varphi \neq \alpha, \neg\alpha$, $N(P',\varphi) = N(P,\varphi)$. Note that in $\geq_{P'}$, α appears either at an earlier position or in the same position as in $\geq_P$. Therefore, if $\succ'$ be an order refining $\geq_{P'}$, when α is considered in $\succ$, it must be consistent with D, otherwise there would be an order $\succ$ refining $\geq_P$ resulting in a subagenda not containing α. Therefore α belongs to all subagendas in $RA(P')$.

3. We consider $R^{d_H,max}$. Consider the pre-agenda $[X] = \{a, b, \underbrace{\neg(a \to b) \vee (a \wedge b)}_{\alpha}\}$[6], and profile P for three agents:

voters	a	$\neg a$	b	$\neg b$	α	$\neg\alpha$
1	−	+	−	+	−	+
2	−	+	+	−	−	+
3	+	−	+	−	+	−

and its b-reinforcement (in the first voter's judgment set) P':

voters	a	$\neg a$	b	$\neg b$	α	$\neg\alpha$
1	−	+	+	−	−	+
2	−	+	+	−	−	+
3	+	−	+	−	+	−

As it can be observed from table 4, $b \in T_{R^{d_H,max}}(P)$, since $R^{d_H,max}(P) = \{\{\neg a, b, \neg\alpha\}\}$.

However, as it can be observed from Table 5, although $b \in T_{R^{d_H,max}}(P')$, it is not the case that $R^{d_H,max}(P) = R^{d_H,max}(P')$ since $R^{d_H,max}(P') = \{\{a,b,\alpha\},\{\neg a, b, \neg\alpha\}\}$.

[6] Observe that α enforces that a holds.

$A \in \Phi_X$	$d_H(A,A_1)$	$d_H(A,A_2)$	$d_H(A,A_3)$	*max*
$\{a,b,\alpha\}$	3	2	0	3
$\{\neg a, b, \neg\alpha\}$	1	0	2	2
$\{a, \neg b, \alpha\}$	2	3	1	3
$\{\neg a, \neg b, \neg\alpha\}$	0	1	3	3

Table 4: The max of Hamming distances from an element in the set Φ_X to each of the agent's judgment sets in profile P.

$A \in \Phi_X$	$d_H(A,A_1')$	$d_H(A,A_2)$	$d_H(A,A_3)$	*max*
$\{a,b,\alpha\}$	2	2	0	2
$\{\neg a, b, \neg\alpha\}$	1	0	2	2
$\{a, \neg b, \alpha\}$	3	3	1	3
$\{\neg a, \neg b, \neg\alpha\}$	0	1	3	3

Table 5: The max of Hamming distances from an element in the set Φ_X to each of the agent's judgment sets in profile P'.

4. We consider R_{MWA} *i.e.*, $R^{d_H,\Sigma}$. Let P be a profile $P = (A_1,\ldots,A_k,\ldots,A_n)$ and its a α-reinforcement, a profile $P' = (A_1',\ldots,A_k',\ldots,A_n') = (A_1,\ldots,A_k^*,\ldots,A_n)$. Let Φ_X be the set of all consistent and complete judgment sets over an agenda X. For any $B \in \Phi_X$, we can write, without the loss of generality that

$$d_H(B,A_1)+,\ldots,+d_H(B,A_k),\ldots,d_H(B,A_n) = K(B) + d_H(B,A_k)$$

and

$$d_H(B,A_1')+,\ldots,+d_H(B,A_k'),\ldots,d_H(B,A_n') = K(B) + d_H(B,A_k^*).$$

If $\alpha \in T_{R^{d_H,max}}(P)$, then for all $A \in R^{d_H,\Sigma}(P)$, we have that $\alpha \in A$.Assume that $\alpha \in T_{R^{d_H,max}}(P)$.

Observe that the distance d_H can be defined as $\sum_{\varphi \in A} h(\varphi, A, A_i)$, where $h(\varphi,A,A_i) = 0$ if $\varphi \in A$ and $\varphi \in A_i$ and $h(\varphi,A,A_i) = 1$ otherwise. Since for all $i \neq k$, $A_i = A_i'$, $d_h(A,A_i) = d_h(A,A_i')$. Since $A_k^* \setminus \{\alpha\} = A_k \setminus \{\neg\alpha\}$, we have that $d_h(A,A_k^*) = d_h(A,A_k) - 1$. Consequently

$$\sum_{j=1}^{n} d_H(A,A_j) = K(A) + d_H(A,A_k) \tag{1}$$

and

$$\sum_{j=1}^{n} d_H(A,A_j') = K(A) + d_H(A,A_k) - 1.$$

We can conclude that the winners for P' have a lower score than the winners for P.

By the definition of $R^{d_H,\Sigma}$, for all $A^\circ \in \Phi_X$, $A^\circ \notin R^{d_H,max}(P)$:

$$\sum_{i=1}^{n} d_H(A,A_i) < \sum_{i=1}^{n} d_H(A^\circ,A_i). \tag{2}$$

Hence

$$K(A) + d_H(A,A_k) < K(A^\circ) + d_H(A^\circ,A_k). \tag{3}$$

Assume that there exists among those A°, one such that $A^\circ \in R^{d_H,\Sigma}(P')$ and $A^\circ \notin R^{d_H,\sigma}(P)$. In this case

$$\sum_{i=1}^{n} d_H(A,A_i') \geq \sum_{i=1}^{n} d_H(A^\circ,A_i'). \tag{4}$$

From (4) it would follow that

$$K(A)+d_H(A,A_k)-1 \geq K(A^\circ)+d_H(A^\circ,A'_k). \quad (5)$$

From (3) and (5) it follows that

$$K(A^\circ)+d_H(A^\circ,A'_k)+1 \leq K(A^\circ)+d_H(A^\circ,A_k). \quad (6)$$

If $\neg\alpha \in A^\circ$ then (6) is impossible (recall that A'_k and A_k only differ on the judgment for α, in that $\alpha \in A'_k$ and $\neg\alpha \in A_k$). If $\alpha \in A^\circ$ then

$$K(A^\circ)+d_H(A^\circ,A'_k)+1 = K(A^\circ)+d_H(A^\circ,A_k).$$

However, since $K(A^\circ)+d_H(A^\circ,A_k) > K(A)+d_H(A,A_k)$, and the winners of $R^{d_H,\Sigma}(P')$ have a lower score than the winners of $R^{d_H,\Sigma}(P)$, it cannot be the case that $A^\circ \in R^{d_H,\Sigma}(P')$. ■

As for R_Y, reinforcement is not satisfied. Consider the agenda $X=\{a,\neg a,b,\neg b,a\wedge b,\neg(a\wedge b),c\}$ and the profile P

Voters	a	$\neg a$	b	$\neg b$	$a\wedge b$	$\neg(a\wedge b)$	c	$\neg c$
1,2	+	−	+	−	+	−	+	−
3,4	+	−	−	+	−	+	+	−
5	+	−	−	+	−	+	+	−
6,7,8,9	−	+	+	−	−	+	−	+

P is not majority consistent and we obtain that $R_Y(P)=\{\{b,\neg(a\wedge b)\}\}$ by removing one of the judgment sets $\{a,\neg b,\neg(a\wedge b),c\}$ from P. Consequently $\{\neg a,b,\neg(a\wedge b)\} \in T_{R_Y}(P)$. Consider now the profile P', which is a $\neg a$-reinforcement on P (in the judgment set of the 5th voter).

Voters	a	$\neg a$	b	$\neg b$	$a\wedge b$	$\neg(a\wedge b)$	c	$\neg c$
1,2	+	−	+	−	+	−	+	−
3,4	+	−	−	+	−	+	+	−

P' is majority consistent and we obtain $R_Y(P)=\{\{\neg a,b,\neg(a\wedge b),r\}\}$. Consequently $\{\neg a,b,\neg(a\wedge b),r\} \in T_{R_Y}(P')$ and $R_Y(P) \neq R_Y(P')$

4.3 Separability

We define a judgment aggregation version of the separability property defined by [23]. The same property is called consistency in [24]. It is also sometimes called *reinforcement*.

Definition 14 (Separability) *For all profiles $P_1,P_2 \in Dom(F)$, with $P_1=\langle A_1,\dots,A_{n_1}\rangle$ and $P_2=\langle B_1,\dots,B_{n_2}\rangle$, we define $P1+P2$ as the n_1+n_2-profile $\langle A_1,\dots,A_{n_1},B_1,\dots,B_{n_2}\rangle$. Then we say that a rule R satisfies* separability *if for every judgment sets P_1,P_2 such that $\alpha \in T_R(P_1)$ and $\alpha \in T_R(P_2)$, then $\alpha \in T_R(P_1 \cup P_2)$.*

Proposition 6 *None of the six aggregation rules R_Y, R_{MSA}, R_{MCSA}, R_{MWA} and R_{RA} and $R^{d_H,max}$ satisfies separability.*

PROOF. A single profile will suffice for the first five rules: let $X=\{p,\neg p,q,\neg q,p\vee q,\neg p\wedge\neg q\}$ and the 10-voter profile as follows:

voters	p	$\neg p$	q	$\neg q$	$p\vee q$	$\neg p\wedge\neg q$
1,2,3	+	−	−	+	+	−
4,5	−	+	−	+	−	+
6,7,8	−	+	+	−	+	−
9,10	−	+	−	+	−	+

Consider also the two subprofiles P_1 consisting of voters 1 to 5 and P_2 consisting of voters 6 to 10. Note first that P_1 and P_2 are majority-consistent, and that their majoritarian aggregation is $m(P_1)=\{p,\neg q,p\vee q\}$ and $m(P_2)=\{q,\neg p,p\vee q\}$. Since the five aggregation rules satisfy majority-consistency, for all $R \in \{R_Y,R_{MSA},R_{MCSA},R_{MCIS},R_{RA}\}$ we have $R(P_1)=\{\{p,\neg q,p\vee q\}\}$ and $R(P_2)=\{\{q,\neg p,p\vee q\}\}$; therefore, $p\vee q \in T_R(P_1)$ and $p\vee q \in T_R(P_2)$. Now, we claim that $p\vee q \notin T_R(P_1\cup P_2)=T_R(P)$.

- R_{MSA}: note first that $m(P)=\{\neg p,\neg q,p\vee q\}$. There are three maximal subagendas for P, namely $\{p,\neg p,q,\neg q\}$, $\{p,\neg p,p\vee q,\neg p\wedge\neg q\}$ and $\{q,\neg q,p\vee q,\neg p\wedge\neg q\}$. Whereas $p\vee q$ is inferred in the latter two, it is not in the first one, therefore $p\vee q \notin T_{R_{MSA}}(P)$.
- R_{MCSA}: same proof, noticing that the two maximal subagendas are also maxcard subagendas.
- R_{MCIS}: we have $N(P,p)=3$, $N(P,\neg p)=7$, $N(P,q)=3$, $N(P,\neg q)=7$, $N(P,p\vee q)=6$ and $N(P,\neg(p\vee q))=4$. The only consistent subagenda of X of maximum weight is $\{\neg p,\neg q,\neg(p\vee q)\}$, therefore $p\vee q \notin T_{R_{MCIS}}(P)$.
- R_{RA}: reordering the elements of X following $\succ$ we get $\neg p$, $\neg q$, $p\vee q$, $\neg p\wedge\neg q$, p, q; therefore $R_{RA}(P)=\{\neg p,\neg q,\neg p\wedge\neg q\}$, and $R_{RA}(P) \not\models p\vee q$.
- R_Y: the maxcard consistent subsets of judgments are all subsets of voters containing $\{4,5,9,10\}$ and exactly two among $\{1,2,3\}$ and two among $\{6,7,8\}$. Therefore $R_Y(P)=\{\{\neg p,\neg q,\neg p\wedge\neg q\}\}$, and $p\vee q \notin T_{R_{RA}}(P)$.

Finally, we consider $R^{d_H,max}$. Consider the pre-agenda $[X]=\{p,q,r,p\rightarrow(q\wedge r)\}$, and the 5-voter profile P:

voters	p	q	r	$p\rightarrow(q\wedge r)$
1,2,3	+	+	−	−
4,5	+	+	+	+

Consider also the two sub-profiles P_1 consisting of voters 1, 2 and 3, and P_2 consisting of voters 4 and 5. Observe that $R^{d_H,max}(P_1)=\{\{p,q,\neg r,\neg(p\rightarrow(q\wedge r))\}\}$ and $R^{d_H,max}(P_2)=\{\{p,q,r,p\rightarrow(q\wedge r)\}\}$, thus $p \in T_{R^{d_H,max}}(P_1)$ and $p \in T_{R^{d_H,max}}(P_2)$. However, $R^{d_H,max}(P)=\{\{p,q,\neg r,\neg(p\rightarrow(q\wedge r))\},\{p,q,r,p\rightarrow(q\wedge r)\},\{\neg p,q,\neg r,p\rightarrow(q\wedge r)\},\{p,\neg q,r,\neg(p\rightarrow(q\wedge r))\}\}$, therefore $p \notin T_{R^{d_H,max}}(P)$. ■

5. CONCLUSION

We have studied six judgment aggregation rules (two of which were not totally new, but had not been studied from the point of views of the properties we considered). Table 5 summarizes the results on the properties considered for the rules. We also considered the possible inclusion relations between the presented rules, which were summarized in Table 1.

With the rules we propose we have not exhausted all the possibilities for judgment aggregation rules. For example, the Young rule R_Y can be refined further; more distance-based rules can be specified by other pairs of distance measure and aggregation function already featured in [11, 18]. The pool of interesting and significant judgment aggregation rule properties is not exhausted either. Lastly, it is vital to consider the computational properties of the rules we introduce, such as complexity of winner determination (see [9]). Another issue that we need to investigate is the strategic aspect of our aggregation rules (as done in judgment aggregation [3], that is, how voters can strategically submit insincere judgment sets in order to induce specific outcomes. These computational and manipulation issues will be addressed in a companion paper.

	Majority Pres.	Weak Unan.	Strong Unan.	Reinf. App. F.	Separ.
R_Y	✓	✓	✓	✓	no
R_{MSA}	✓	✓	no	✓	no
R_{MCSA}	✓	no	no	✓	no
R_{MWA}	✓	no	no	✓	no
R_{RA}	✓	✓	✓	✓	no
$R^{d_H,max}$	no	no	no	no	no

Table 6: Summary of the results for the social theoretic properties of the judgment aggregation rules.

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