Agreeing to Disagree Type Results under Ambiguity

[Extended Abstract] *

Adam Dominiak
University of Heidelberg
Alfred Weber Institute
Bergheimer Str. 58
69115 Heidelberg, Germany
adam.dominiak@awi.uni-heidelberg.de

Jean Philippe Lefort
University of Heidelberg
Alfred Weber Institute
Bergheimer Str. 58
69115 Heidelberg, Germany
jplefort1@yahoo.fr

ABSTRACT

In this paper we show that unlike in Bayesian frameworks asymmetric information does matter and can explain differences in common knowledge decisions due to ambiguous character of agents' private information. Agents share a common, but-not-necessarily-additive, prior beliefs represented by capacities. It is shown that, if each agent's information partition is made up of unambiguous events in the sense of Nehring [12, *Mat. Soc. Sci.* **38**, 197-213], then it is impossible that the agents disagree on their commonly known decisions, whatever these decisions are: whether posterior beliefs or conditional expectations. Conversely, an agreement on conditionalexpectations, but not on posterior beliefs, implies that agents' private information must consist of Nehring-unambiguous events. The results obtained allow to attribute the existence of a speculative trade to the presence of agents' diverse and ambiguous information.

Categories and Subject Descriptors

G3. [**Probability and Statistics**]: Miscellaneous; H.1.1 [**Systems and Information Theory**]: Value of information, Miscellaneous; I.2.3 [**Deduction and Theorem Proving**]: Uncertainty, "fuzzy," and probabilistic reasoning

General Terms

Theory

Keywords

Ambiguity, capacities, Choquet expected utility theory, unambiguous events, asymmetric information, common knowledge, agreement theorem.

1. INTRODUCTION

In his celebrated article on "Agreeing to Disagree", [1] challenged the role that asymmetric information plays in the context of interpersonal decision problems under uncertainty. Aumann showed that, if two agents share a common prior probability distribution, and their posterior beliefs for some fixed event are common knowledge, then these posteriors must coincide, although they may be conditioned on diverse information. Aumann's result has been extended in a number of directions. For instance, [7] showed that the agents can neither disagree on the values of common knowledge posterior expectations. [10] proved that the only common knowledge posterior trade, among the agents with the same risk attitude, is the zero trade. All these results suggested that within Bayesian frameworks, combined with the doctrine of common priors, asymmetric information has less explanatory power than has been thought: Differences in decisions cannot be explained *solely* on the basis of asymmetric information. In this paper we show that asymmetric information does matter and can explain differences in common knowledge decisions due to *ambiguous* character of agents' private information.

In order to explain differences in commonly known decisions, [11] advocated to weaken the "commonness" assumption of prior beliefs, while still assuming that the agents are subjective expected utility maximizer. Here, we suggest an alternative approach. We maintain the assumption of common prior beliefs, but weaken the "additivity" requirement by allowing the agents to be Choquet expected utility maximizer in the spirit of [13]. According to Schmeidler's theory subjective beliefs are represented by a normalized and monotone, but-non-necessarily-additive, set function, called a *capacity*. The notion of capacity allow ambiguity and different ambiguity attitudes to play role when making decision. In the presence of non-additive beliefs expected utilities are computed by means of Choquet integrals.

Throughout, we assume that the agents share a common capacity distribution over an algebra of events generated by a finite set of states. Furthermore, each agent is characterized by a partition over the set of states which represents her private information. There are two stages: ex-ante and ex-post. At ex-ante stage the agents share identical information. At ex-post stage, the agents receive private signals, they incorporate the receipt of new information by updating their prior beliefs conditional on their private information and then they announce their decisions. We consider two types of decisions: posterior capacities for some fixed event and conditional Choquet expectations of some fixed action.

*A full version of this paper is available upon request

Our first objective is to characterize the properties of events in each agent's information partition which guarantee that it is impossible that the agents disagree on their common knowledge decisions. A natural candidate for such events are those events for which probabilistic information is given. Recently, several notions of unambiguous events have been suggested, e.g. by [12] and by [14]. In Bayesian frameworks all uncertain events are unambiguous. In non-Bayesian setups, however, only some of them may be unambiguous. We start our analysis by assuming that each agent's information partition consists of unambiguous events, while other events may be ambiguous. It turns out that, whenever each agent's information partition is made up of unambiguous events in the sense of [12,] then it is impossible that they disagree on their commonly known decisions, whatever these decisions are, whether posterior capacities or conditional Choquet expectations. Furthermore, it is shown that even a small departure from Nehring's notion of unambiguous events may create an opportunity for a disagreement. That is, by adopting a weaker notion of unambiguous events, as proposed by [14], it is shown that at some state agent's posterior capacities for some fixed event are common knowledge, but they are not the same.

As next we focus on a converse result. It turns out that when observing an agreement in posterior beliefs or in posterior expectations for some fixed binary action, nothing can be said about the properties of events in agents' information partitions. Motivated by these observations we consider the set of more general actions. It is shown that an agreement on conditional Choquet expectations for some many-valued action implies that each agents' private information is made up of N-unambiguous events.

Based on the previous results, we generalize the no-trade theorem of [10] in the context of ambiguity. It is shown that, whenever each agent's information partition is made up of Nehring-unambiguous events then the only trade, for which it is common knowledge among the Choquet expected utility maximizers that each of them is willing to engage in it, is the zero trade. Thus, ambiguity may be seen as offering an intuitive explanation for the widely observed speculative activities in the real markets.

2. SETUP

2.1 Knowledge Structure

We consider a finite set Ω of states. An event E is a subset of Ω. Let $\mathcal{A} = 2^\Omega$ be the set of all subsets of Ω. For any $E \subset \Omega$ we denote $\Omega \setminus E$, the complement of E, by E^c. There is a finite group of agents I indexed by $i = 1, \ldots, N$. Each agent i is endowed with a partition $\mathcal{P}_i$ of Ω, which represents i's private information in the usual sense. It is said that the agent i knows an event E at ω if $\mathcal{P}_i(\omega) \subset E$. The event that i knows E, denoted by $K_i E$, is a set of all states in which i knows E, i.e.:

$$K_i E = \{\omega \in \Omega : \mathcal{P}_i(\omega) \subset E\}. \tag{1}$$

An event E is common knowledge at a state ω if everyone knows E at ω, everyone knows that everyone knows E at ω, and so on, ad infinitum. Let $\mathcal{M} = \wedge_{i=1}^N \mathcal{P}_i$ be the meet (i.e. finest common coarsening) of all agents' partitions. Denote by $\mathcal{M}(\omega)$ the member of $\mathcal{M}$ that contains ω. Then, E is commonly known at ω if and only if $\mathcal{M}(\omega) \subset E$ (e.g., see [1]).

2.2 Static Choquet Preferences

According to Choquet expected utility theory of [13] agents' beliefs are represented by *capacities*. A capacity $\nu : \mathcal{A} \to \mathbb{R}$ is a set function such that *i)* $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$ and *ii)* $\nu(E) \leq \nu(F)$ whenever $E \subset F \subset \Omega$. Let X be a set of outcomes. Actions $f : \Omega \to X$ are mappings from states to outcomes. Let $\mathcal{F}$ be a set of all actions. A preference relation $\succcurlyeq$ over $\mathcal{F}$ admits *Choquet expected utility* representation if there exists a utility function u and a capacity ν such that for any $f, g \in \mathcal{F}$:

$$f \succcurlyeq g \Leftrightarrow \int_\Omega u \circ f \, d\nu \geq \int_\Omega u \circ g \, d\nu. \tag{2}$$

The expectations are taken in the following sense. For a given action f let $\rho : \{1, \ldots, n\} \to \Omega$ be a bijection which orders the partition $E_{\rho(1)}, \ldots, E_{\rho(n)}$ from most to least favorable events under f, i.e. $u(x_{\rho(1)}) \geq \cdots \geq u(x_{\rho(n)})$. The bijection ρ expresses the ranking position of events associated with f. The *Choquet integral* of f with respect to ν and u is defined as:

$$\int_\Omega u \circ f \, d\nu =$$

$$\sum_{j=1}^{n-1} [u(x_{\rho(j)}) - u(x_{\rho(j+1)})] \, \nu(E_{\rho(1)}, \ldots, E_{\rho(j)}) + u(x_{\rho(n)}) \tag{3}$$

For a given capacity ν and ρ define a *rank-dependent probability* distribution p_ρ^ν on Ω as:

$$p_\rho^\nu(E_{\rho(j)}) = \nu(E_{\rho(1)}, \ldots, E_{\rho(j)}) - \nu(E_{\rho(1)}, \ldots, E_{\rho(j-1)}). \tag{4}$$

Then, (3) can be equivalently written as Choquet integral of f with respect to the rank-dependent probability distribution p_ρ^ν and u, i.e.:

$$\int_\Omega u \circ f \, dp_\rho^\nu = \sum_{j=1}^{n-1} [u(x_{\rho(j)})] \, p_\rho^\nu(E_{\rho(j)}) \tag{5}$$

In the context of ambiguity it is important to localize events that are unambiguous, i.e. events on which agents have some kind of probabilistic beliefs. [12] associate ambiguity of events with their rank dependence. In particular, he calls an event U unambiguous, henceforth N-unambiguous, if the probability attached to the event does not depend the ranking position of U. Let $\mathcal{R}$ be a set of all possible ranks. Accordingly, an event $U \in \mathcal{A}$ is called N-unambiguous if $p_\rho^\nu(U) = \nu(U)$ for all $\rho \in \mathcal{R}$. Let $\mathcal{A}_N^U$ be the collection of all N-unambiguous events. [12] showed that $\mathcal{A}_N^U$ is always an algebra. Furthermore, he showed that $U \in \mathcal{A}_N^U$ if and only if for all $E \in \mathcal{A}$:

$$\nu(E) = \nu(E \cap U) + \nu(E \cap U^c). \tag{6}$$

[14] suggested an alternative definition of unambiguous events. He calls an event U unambiguous, henceforth Z-unambiguous, if and only if for all $E \in \mathcal{A}$ such that $E \subset U^c$:

$$\nu(E \cup U) = \nu(E) + \nu(U). \tag{7}$$

Let $\mathcal{A}_Z^U$ be the collection of all Z-unambiguous events. It is worth to mention that $\mathcal{A}_N^U \subset \mathcal{A}_Z^U$, since $\mathcal{A}_Z^U$ does not need to be an algebra.

2.3 Updating Choquet Preferences

At interim stage agents are informed that the true state is ω, they revise their beliefs conditional on $\mathcal{P}_i(\omega)$ and construct conditional preferences with regard to their new beliefs. Throughout, we assume that each agent's private information consists of non-null events, that is $\nu(\mathcal{P}_i(\omega)) > 0$ for all states $\omega \in \Omega$ and for all $i \in I$. If the true state is ω, we denote by $\succcurlyeq_{\mathcal{P}_i(\omega)}$ agent i's conditional CEU preferences over $\mathcal{F}$ given her information $\mathcal{P}_i(\omega)$, i.e. for all $f, g \in \mathcal{F}$:

$$f \succcurlyeq_{\mathcal{P}_i(\omega)} g \iff$$

$$\int_\Omega u \circ f \; d\nu(\cdot \mid \mathcal{P}_i(\omega)) \geq \int_\Omega u \circ g \; d\nu(\cdot \mid \mathcal{P}_i(\omega)), \qquad (8)$$

where $\nu(\cdot \mid \mathcal{P}_i(\omega))$ is a capacity conditional on $\mathcal{P}_i(\omega)$. In the class of probabilistic models agents revise their beliefs by using the Bayes rule. In non-probabilistic models there are many possible ways how to update beliefs (see [8]). Beside the Bayes rule there are two common revision rules for capacities: the Maximum-Likelihood updating rule, introduced by [4], and the Full-Bayesian updating rule suggested by [9]. In [5] it has been shown that when updating on N-unambiguous events then the Maximum-Likelihood updating rule as well as with the Full-Bayesian updating rule coincides with the Bayes rule.

3. AGREEMENT THEOREMS

3.1 Sufficient Condition

Let $\mathcal{D}$ be a non-empty set of possible decisions. Decisions are determined by i's *decision function* $d_i : \Omega \to \mathcal{D}$ which is a function of i's private information, i.e. $d_i(\omega) = d_i(\mathcal{P}_i(\omega))$. We mainly consider two types of decision functions: i) a conditional capacity for some event $E \in \mathcal{A}$,

$$d_i(\omega) = \nu(E \mid \mathcal{P}_i(\omega)), \qquad (9)$$

or ii) a conditional Choquet expectation for some action $f \in \mathcal{F}$,

$$d_i(\omega) = \int_\Omega u \circ f \; d\nu(\cdot \mid \mathcal{P}_i(\omega)). \qquad (10)$$

Let $D_i(\xi_i) = \{\omega : d(\mathcal{P}_i(\omega)) = \xi_i\}$ be the event that the agent i makes a decision ξ_i. Essentially, an agreement theorem states that if at some state agents' decisions are common knowledge then they must be the same the same. Formally, if at some state ω^* the event $\bigcap_{i\in I} D_i(\xi_i)$ is common knowledge, i.e. $\mathcal{M}(\omega^*) \subset \bigcap_{i\in I} D_i(\xi_i)$, then $\xi_1 = \xi_2 = \cdots = \xi_N$. When the decision function $d_i(\cdot)$ is defined as a conditional probability (9) and the agreement theorem holds true, we designate this situation as an *Agreement in Beliefs*. When the decision function $d_i(\cdot)$ is defined as a conditional expectation (10) and the agreement theorem holds true, we term this situation an *Agreement in Expectations*.

In the existing non-Bayesian extensions of probabilistic agreement theorems, established for instance by [2], the nature of agents' subjective beliefs is inessential and the decision function may be an arbitrary function. To guarantee that the agreement theorem holds it is required, that agents are "like-minded" and that the decision function $d(\cdot)$ satisfies the *Sure-Thing-Condition* (STC), i.e. for any partition $E_1, \ldots, E_n$ of Ω it is true that:

$$d_i(E_1) = \cdots = d_i(E_n) = \xi_i \quad \Rightarrow \quad d(\bigcup_{j=1}^{n} E_j) = \xi_i. \qquad (11)$$

Note, in the class of probabilistic models, decision functions such as conditional probabilities as well as conditional expectations satisfy the STC on *any* partition. In non-probabilistic models, however, the decision function may satisfy the STC on some fixed partitions, but not on others.

For this reason our first objective is to fix a partition and to look at properties of events of that partition which are sufficient for a decision function $d(\cdot)$ to satisfy the STC on it. It turns out that the decision function $d_i(\cdot)$, whether it is a conditional capacity or a conditional Choquet expectation, satisfies STC on a fixed partition whenever this partition is made up of N-unambiguous events. This condition on its own is a sufficient condition for agreement theorem to hold under ambiguity. That is, if each agent i's private information is represented by a partition $\mathcal{P}_i$ made up of N-unambiguous events, then the agents cannot disagree on their commonly known decisions, whatever these decisions are: whether conditional capacities or conditional Choquet expectations.

THEOREM 1. *Let ν be a common capacity distribution on Ω and let $\mathcal{A}_N^U \subset \mathcal{A}$ be a collection of N-unambiguous events. Let $P_1^i, \ldots, P_k^i, \ldots, P_K^i$ be the events in i's partition $\mathcal{P}_i$. If $P_k^i \in \mathcal{A}_N^U$ for all $k = 1, \ldots, K$ and all agents $i \in I$, then the following statements are true:*

(i) Agreement in Beliefs holds,

(ii) Agreement in Expectations holds.

How strong is the sufficiency condition in Theorem 1? In particular, suppose that we adapt a weaker notion of unambiguous events, for instance, the one proposed by [14]. Is the claim still true that a disagreement in commonly known decisions is impossible? Example 1 answers this question negatively. Even a small departure from Nehring's notion of unambiguous events may create disagreement opportunities.

EXAMPLE 1. *Consider two agents $I = \{A, B\}$, called Anna and Bob, the set of states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, the set of decisions $\mathcal{D} = [0, 1]$ and the decision function defined as in (9). Let $\mathcal{P}_A = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{P}_B = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the agents' information partitions. Anna and Bob face the following capacity distribution on $\mathcal{A}$:*

$$\begin{array}{lll}
\nu(\omega_j) = \frac{1}{10}, & \textit{for any} & j = 1, \ldots, 4, \\
\nu(\omega_j, \omega_k) = \frac{1}{2}, & \textit{for any} & j + k \neq 5, \\
\nu(\omega_j, \omega_k) = \alpha, & \textit{for any} & j + k = 5 \textit{ where } \alpha \in [\frac{1}{10}; \frac{1}{2}), \\
\nu(\omega_j, \omega_k, \omega_l) = \frac{6}{10}, & \textit{for any} & j, k, l = 1, \ldots, 4.
\end{array}$$

Note, all events $\{\omega_j, \omega_k\}$ with $j+k \neq 5$ are Z-unambiguous, but not N-unambiguous. To see this, consider the event $\{\omega_1, \omega_2\}$ and its complement $\{\omega_3, \omega_4\}$. On this partition the capacity sums up to one. Now, if these events were N-unambiguous, then according to (6) the capacity for the event $\{\omega_1, \omega_3\}$ were $\nu(\omega_1, \omega_3) = \nu(\omega_1) + \nu(\omega_3) = \frac{1}{5}$, but not $\frac{1}{2}$. One can verify that the capacity ν satisfies (7). Accordingly, $\mathcal{A}_Z^U = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}, \Omega\}$ is the collection of Z-unambiguous events.

Therefore, Anna's partition is made up of Z-unambiguous events which are not N-unambiguous. Consider the event $E = \{\omega_1, \omega_3\}$. At any state Anna and Bob announce their posterior beliefs for the occurrence of E given their private information. Given Bob's private information he announces $d_B(\omega) = \nu(E \mid \mathcal{P}_B(\omega)) = \frac{1}{2}$ at any state $\omega \in \Omega$. Anna has finer information than Bob and therefore she announces $d_A(\omega) = \nu(E \mid \mathcal{P}_A(\omega)) = \frac{1}{5}$ for all $\omega \in \Omega$. Note, Anna's decision function $d_A(\cdot)$ violates the Sure-Thing Condition on her partition. Furthermore, since $\mathcal{M} = \Omega$, the event that Anna's decision is $\frac{1}{5}$ and that Bob's decision is $\frac{1}{2}$ is commonly known at any state. But, these decisions are in fact not the same. This shows that, if for one agent her private information is made up of Z-unambiguous events, which are not N-unambiguous, than the STC is violated and it is possible that the agents end up with agreeing to disagree!

3.2 The Converse Result

In this section we address the following issue. Suppose that agents' decisions satisfy the STC on their information partitions and that the agents cannot disagree on their commonly known decisions. Can we infer something about the nature of agents' private information? In principle, the answer is "Yes". However, what we may infer observing an agreement depends on the type of decisions on which agents agree to agree. There are situations in which Agreement in Beliefs is present and nothing can be said about the nature of agents' private information. The reason is the following one. We can easily find a capacity distribution and an updating rule such that at some state an agreement on posterior beliefs for some fixed event holds true and the agents' information partitions are neither made up of N-unambiguous nor of Z-unambiguous events. For instance, [6] characterize the family of updating rules for neo-additive capacities, axiomatized by [3], which are necessary and sufficient for Aumann's agreement theorem to hold in the context of such beliefs. However, for this type of capacities, by construction, it is impossible that only some sub-algebra of events is unambiguous (the only possible unambiguous algebra is $\mathcal{A}$ as in the probabilistic case).

Furthermore, it turns out that if at some state ω it impossible that the agents agree to disagree on conditional capacities for some event E then it is also impossible at ω that they agree to disagree on conditional Choquet expectations of binary actions defined on the event E. A binary action $b = x_E y$ is a function which assigns the constant outcome $f(\omega) = x \in X$ to each state ω in E and the constant outcome $f(\omega) = y \in X$ to each ω in E^c.

PROPOSITION 1. *Let ν be a common capacity distribution ν on Ω. Let $\mathcal{P}_i$ be i's information partition and let $d_i(\cdot)$ be i's conditional capacity for some event $E \in \mathcal{A}$. Suppose that at some state ω^* Agreement in Beliefs holds true. Consider a binary action $b = x_E y$ defined on the event E. Let $\tilde{d}_i$ be i's conditional Choquet expectation of b. Then, Agreement in Expectations holds true at ω^*.*

Thus, for the same reason as above, nothing can be said about the nature of events representing agents private information when knowing that the agents cannot agree to disagree on expectations for some binary action.

For this reason we constrain our attention to the whole set of possible actions $\mathcal{F}$ and ask again whether it is possible to infer something about the nature of events in an agent's partition knowing that the agents reached Agreement in Expectations for more general action f. Theorem 2 answers this question in the affirmative. Agreement in Expectations for a many-valued action implies that agents' information partitions are made up of N-unambiguous events.

THEOREM 2. *Let ν be a common capacity distribution on Ω. Let $\mathcal{A}'$ be a sub-algebra of $\mathcal{A}$. Let $d_i(\cdot)$ be the Choquet conditional expectation for some many-valued action f in $\mathcal{F}$. If for any information partition $\mathcal{P}_i = P_1^i, \ldots, P_k^i, \ldots, P_K^i$ such that $P_k^i \in \mathcal{A}'$ for all $k = 1, \ldots, K$ and all agents $i \in I$, $d_i(\cdot)$ satisfies the STC on $\mathcal{P}_i$, then $\mathcal{A}'$ is the algebra made up of N-unambiguous events.*

4. REFERENCES

[1] R. J. Aumann. Agreeing to disagree. *The Annals of Statistics*, 4:1236–1239, 1976.
[2] M. Bacharach. Some extension of a claim of aumann in axiomatic model of knowledge. *Journal of Economic Theory*, 37:167–190, 1985.
[3] A. Chateauneuf, J. Eichberger, and S. Grant. Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *Journal of Economic Theory*, 137:538–567, 2007.
[4] A. P. Dempster. A generalization of bayesian inference. *Journal of Royal Statistical Society Series B*, 30:205–247, 1968.
[5] A. Dominiak and J.-P. Lefort. Unambiguous events and dynamic choquet preferences. *Economic Theory*, 2009. forthcoming.
[6] A. Dominiak and J.-P. Lefort. Agreement thoerem for neo-additive beliefs. Mimeo, University of Heidelberg, 2010.
[7] J. Geanakoplos and J. Sebenius. Don't bet on it: Contingent agreements with asymmetric information. *Journal of American Statistical Association*, 78:424–426, 1983.
[8] I. Gilboa and D. Schmeidler. Updating ambiguous beliefs. *Journal of Economic Theory*, 59:33–49, 1993.
[9] J. Jaffray. Bayesian updating and belief functions. *IEEE Transactions on Systems, Man and Cybernetics*, 22:1144–1152, 1992.
[10] P. Milgrom and N. Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26:17–27, 1982.
[11] S. Morris. Trade with heterogeneous prior beliefs and asymmetric information. *Econometrica*, 62:1327–1347,

1994.

[12] K. Nehring. Capacities and probabilistic beliefs: A precarious coexistence. *Mathematical Social Science*, 38:197–213, 1999.

[13] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

[14] J. Zhang. Subjective ambiguity, expected utility and choquet expected utility. *Economic Theory*, 20:159–181, 2002.