Complexity Results for Logics of Local Reasoning and Inconsistent Belief

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Abstract

Fagin, Halpern, Moses, and Vardi have proposed a framework of epistemic agents with multiple "frames of mind" (*local-reasoning structures*), to solve problems concerning inconsistent knowledge and logical omniscience. We investigate a class of related modal logics. These logics replace the usual closure under full conjunction for the $\Box$ operator with progressively weaker versions, and comprise a hierarchy with the traditional modal logic K at the top, and an infinite number of logics ordered by inclusion under it, all strictly stronger than N, the weakest monotonic modal logic. Previous results have used N to represent local-reasoning structures. Our result shows that there are stronger logics applicable to such structures, suggesting that stronger forms of inference can be used to represent imperfect knowledge-based agents and protocols. Further, it is shown that the satisfiability question for each of these logics is PSPACE-complete, strictly harder than for N. This also answers a conjecture of Vardi: the border between NP- and PSPACE-hardness in modal-logical satisfiability problems is associated with conjunctive closure, however weak.

1 Introduction

Fagin, Halpern, Moses, and Vardi [5] have proposed the idea of "local reasoning," both as a possible solution to the logical omniscience problem in epistemic logic, and as a model for knowledge-based agents and protocols dealing with inconsistent beliefs or information. We examine a class of modal logics that provide stronger systems of inference for local-reasoning structures than those previously considered. Furthermore, we prove that satisfiability for these logics is PSPACE-complete, establishing a new theoretical result establishing the boundary between the complexity classes NP and PSPACE for modal logics, depending upon the presence of *conjunctive closure* principles.

In the typical Kripke semantics for epistemic logics, agent a_i in state s knows proposition B (written $\Box_i B$) if and only if B holds at every state s' accessible from s via binary relation $R_i(x, y)$. Since "states" in this semantics are maximal and consistent sets of propositions, the contents of a_i's knowledge can only be inconsistent if relation R_i is empty, and a_i holds *all possible beliefs*, since the knowledge-condition is vacuously satisfied. Only then can, for example, both of the formulae $\Box_i B$ and $\Box_i \neg B$ hold for a_i at s.

In local-reasoning semantics, on the other hand, each agent a_i has "multiple frames of mind": a function C_i over states, where $C_i(s) = \{T_1, \ldots, T_n\}$, and each T_j is a distinct set of states. In effect, each such set functions as its own accessibility relation, one for each possible state of mind of the agent a_i. The semantics for the knowledge operator $\Box_i$ is adjusted to reflect this change: $\Box_i B$ is true at point s if and only if B holds at every $s' \in T_j$, for *some* $T_j \in C_i(s)$. Although each point s' must still be internally consistent, both $\Box_i B$ and $\Box_i \neg B$ may hold at s, since each of B, $\neg B$ can hold in distinct frames of mind $T_1, T_2 \in C_i(s)$ for a_i, even though there is no point s' at which $(B \wedge \neg B)$ holds.

Local-reasoning semantics thus does away with *conjunctive closure*: no longer do $\Box_i B$ and $\Box_i B'$ imply $\Box_i(B \wedge B')$. The corresponding sound and complete axiomatization is given by the multi-agent extensions of the modal logic commonly known as N (described below). Satisfiability for these logics is NP-complete. On the other hand, satisfiability for the stronger logic K—sound and complete for the class of all Kripke frames, and with full conjunctive closure—is PSPACE-complete [12, 7]. Vardi [21] conjectures that the absence of strong conjunctive closure accounts for this difference, hypothesizing that any logic below K lacking that principle falls into NP rather than PSPACE.

We show that the conjecture in fact fails for an infinite hierarchy of logics K^p ($p \in \mathbb{Z}^+$), each weaker than K. We show a 1-to-1 correspondence between Kripke models for any K^p and local-reasoning structures for exactly p distinct frames of mind. It follows that K^p is sound and complete with respect to such structures. We also show that K^p-SAT is PSPACE-complete, despite the fact that for every $p > 1$, K^p lacks strong conjunctive closure for $\Box$.

K^p logics are interesting for three reasons. First, they are stronger than N, and are useful for reasoning about agents with multiple frames of mind. Second, they are related to paraconsistent logics useful for reasoning in the presence of inconsistent input. Third, the PSPACE-completeness of K^p-SAT answers Vardi's conjecture: strong conjunctive closure is not necessary for the jump from NP to PSPACE; rather, *any degree* of closure does the trick.

2 The K^p Hierarchy

The modal system N is given by all tautologies of propositional logic (PL), and three elementary axiom schemata, for a total of four basic rules:

$$[RPL]: \ \models_{PL} A \ \Rightarrow \models A. \tag{1}$$

$$[MP]: \ \models A \ \& \ \models A \rightarrow B \ \Rightarrow \models B. \tag{2}$$

$$[RM]: \ \models A \rightarrow B \ \Rightarrow \models \Box A \rightarrow \Box B. \tag{3}$$

$$[RN]: \ \models A \ \Rightarrow \models \Box A. \tag{4}$$

Vardi [21] proves that the satisfiability problem N-SAT is in NP. The logic K comprises the same rules as N, with an additional *conjunctive closure* axiom:

$$[K]: \ \models \Box A \wedge \Box B \ \rightarrow \ \Box(A \wedge B). \tag{5}$$

In epistemic logic, this axiom specifies that agents know the conjunction of any two things already known. It is well-established that K is sound and complete for the class of all Kripke frames [3, for example]. Ladner [12] showed that satisfiability for K is PSPACE-complete; Halpern and Moses [7] extend this result to the multi-agent logics K_n, with multiple distinct knowledge-operators $\Box_1, \ldots, \Box_n$.[1] After considering various combinations of the rules and axioms (1)–(5), among others, Vardi [21] conjectures that the dividing line between those logics L for which L-SAT $\in$ PSPACE and those for which L-SAT $\in$ NP is the closure axiom $[K]$. For any logic $L \subseteq S5$ without $[K]$, he hypothesizes, L-SAT $\in$ NP. We establish that in fact weaker principles than $[K]$ suffice to place a logic in PSPACE.

2.1 Extending Kripke Semantics

In the standard Kripke semantics for modal logic, a model consists of a set of points U, relation R between point-pairs (u, v), and propositional valuation V over U. Schotch and Jennings [16, 9] extend this idea, allowing R to hold between points and *p-tuples* of points.

[1] In our results here, we consider only single-agent epistemic logics, with one un-indexed knowledge-operator $\Box$. This is for convenience alone; the multi-agent extension of K^p is straightforward, and all results given extend directly to any such logic.

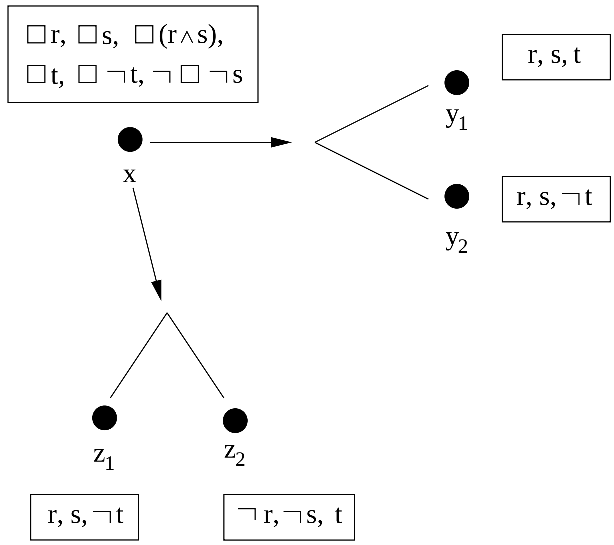

Figure 1: A simple model with a 3-ary accessibility relation. Some formulae true at each point are identified.

Definition 1. Let U be a non-empty set, and $R \subseteq U^{p+1}$. Then $\mathfrak{F} = (U, R)$ is a $(p+1)$-*ary frame*. If $\mathfrak{F}$ is a $(p+1)$-ary frame and V is a valuation function from propositional variables to subsets of U, then $\mathfrak{M} = (\mathfrak{F}, V)$ is a $(p+1)$-ary relational *model*. *Truth* at points in a model $(\mathfrak{M}, x)$ is defined as usual for Boolean operators, and for $\Box$ formulae:

$$(\mathfrak{M},x) \vDash \Box A \quad \text{iff:} \quad (\forall(y_1,\ldots,y_p))\ R(x,y_1,\ldots,y_p) \Rightarrow (\exists i)\ (\mathfrak{M},y_i) \vDash A. \tag{6}$$

That is, $\Box A$ holds at point x just in case A holds at some member of every p-tuple related to x. Figure 1 shows a simple example of a model with a 3-ary accessibility relation; at point x, for instance, $\Box t$ and $\Box\neg t$ are both true, since each of t and $\neg t$ is true at some point in each related pair; their conjunction is not satisfied, however, since there is no related point at which $(t \wedge \neg t)$ holds. The formula $\Box(r \wedge s)$ is satisfied, on the other hand, since both r and s hold at at least one point in each related pair. Lastly, note that $\neg\Box\neg s$ holds at x, since $\neg s$ fails at both members of pair (y_1, y_2).

As p grows, the class of $(p+1)$-ary relational frames corresponds to a modal logic with a progressively weaker conjunctive closure condition. To be precise, the logic K^p is formed by combining the axioms of N [(1)–(4)] with:

$$[K^p] : \vDash (\Box A_1 \wedge \Box A_2 \wedge \cdots \wedge \Box A_{p+1}) \rightarrow \Box \bigvee_{\substack{i,j\in[p+1] \\ (i\neq j)}} A_i \wedge A_j. \tag{7}$$

Note that axiom $[K^1]$ is simply $[K]$ (eq. (5)), and logic $K^1 = K$. For $p > 1$, weaker conjunctive closure holds, e.g.:

$$[K^2] : \vDash (\Box A_1 \wedge \Box A_2 \wedge \Box A_3) \rightarrow \Box((A_1 \wedge A_2) \vee (A_1 \wedge A_3) \vee (A_2 \wedge A_3)).$$

$$[K^3] : \vDash (\Box A_1 \wedge \Box A_2 \wedge \Box A_3 \wedge \Box A_4) \rightarrow \Box((A_1 \wedge A_2) \vee (A_1 \wedge A_3) \vee (A_1 \wedge A_4) \vee (A_2 \wedge A_3) \vee (A_2 \wedge A_4) \vee (A_3 \wedge A_4)).$$

It has been shown that for any value $p \in \mathbb{Z}^+$, K^p is sound and complete for $(p+1)$-ary frames [2, 13]. Further, it is easily seen that these logics form a hierarchy of inclusion such that for any p, $K^{p+1} < K^p$ (i.e., the theorems of K^{p+1} form a proper subset of the theorems of K^p). Finally, it has been proven that system N is the intersection of the denumerable sequence of K^p systems [9]. In a sense, then, this hierarchy comprises all of the "degrees of conjunctive closure" between K proper and N; while K has full closure under conjunction for the $\Box$ operator (given by eq. (5)), and N has no such closure at all, the K^p hierarchy constitutes all the various degrees to which *some* closure is allowed.

2.2 K^p Logics and "Frames of Mind"

We begin by applying the logics in the K^p-hierarchy to the "local reasoning" proposal of Fagin, Halpern, Moses, and Vardi [5, 20, 21]. We strengthen a related soundness and completeness result, by showing that each logic in the K^p hierarchy is sound and complete for a specific class of local-reasoning structures.

Local-reasoning structures have been put forward as a possible solution to the logical omniscience problem in epistemic logic, and as a model for knowledge-based agents and protocols dealing with inconsistent beliefs or information. In the typical Kripke-style semantics for epistemic logics, the knowledge operator is simply the traditional necessity operator, $\Box$. Agent a_i in state s knows proposition B (written $\Box_i B$) if and only if B holds at every state s' accessible from s via the binary relation $R_i(x, y)$. Intuitively, we can understand R_i as holding of a pair of states (s, s') if and only if agent a_i considers state s' possible when in state s. Since "states" in this semantics are maximal and consistent sets of propositions, the contents of a_i's knowledge in any state s can only be inconsistent if relation R_i is empty at s, and a_i holds *all possible beliefs*, since the knowledge-condition is vacuously satisfied. Only then can, for example, both of the formulae $\Box_i B$ and $\Box_i \neg B$ hold for a_i at s.

In local-reasoning semantics, on the other hand, each agent a_i has "multiple frames of mind": a function C_i over states, where $C_i(s) = \{T_1, \ldots, T_n\}$, and each $T_j \subseteq S$ is a distinct set of states. In effect, each such set functions as its own accessibility relation, a separate set of states considered possible relative to s, one for each of a_i's possible states of mind. The semantics for the knowledge operator $\Box_i$ is adjusted to reflect this change: $\Box_i B$ is true at point s if and only if B holds at every state s' contained in *some* frame of mind $T_j \in C_i(s)$. Thus, although each such state s' must still be internally consistent, both $\Box_i B$ and $\Box_i \neg B$ may hold at s, since each of B, $\neg B$ can hold in distinct frames of mind $T_1, T_2 \in C_i(s)$ for a_i, even though there is no point s' at which the explicitly contradictory conjunction $(B \wedge \neg B)$ holds.

In other words, the local-reasoning semantics—like that for the logics in the K^p hierarchy below K proper—eliminates *conjunctive closure*: no longer does $\Box_i B$ and $\Box_i B'$ imply $\Box_i(B \wedge B')$, for any propositions, B, B'. It is known that logic N, weaker than any of the logics K^p, is sound and complete for the class of all local-reasoning structures in which agents may have unboundedly many frames of mind [5, §9.6]. Other authors have considered connections between the K^p modal logics and *paraconsistent logics* applicable to the problem of reasoning and knowledge in the presence of internally-consistent but mutually conflicting data-streams [8, 10, 15]. Here, we extend the use of K^p logics in epistemic contexts, showing how each corresponds to the local-reasoning logic of agents with up to p frames of mind. Here, and throughout the paper, we consider the case of a single agent, for the sake of convenience alone; multi-agent extensions add no real complications.

Definition 2. A *local-reasoning structure for p frames of mind* is a triple $\mathfrak{L}^p = (S, V, C^p)$, where:

1. S is a set of states.
2. $V : Prop \to 2^S$ is a propositional valuation, returning the set of states at which each propositional variable is satisfied.

3. $C^p : S \to (2^S)^p$ is a function from states to p-tuples $(T_1, \ldots, T_p)$ of "frames of mind," each of which is a set of states $T_i \subseteq S$.

Note that we do not require that the various frames of mind, $T_i \in C^p(s)$, be distinct. That is, an agent may not have p different frames of mind, but rather any number n $(1 \leq n \leq p)$ such frames (by duplication where necessary), and in fact the value of n can vary from point to point. Further, if we were to extend this to the multi-agent context, it would be possible to have distinct agents with distinct numbers of frames of mind at one point. These structures thus give us a flexible way of representing interactions between agents whose knowledge may be more or less imperfect in various states of their environment. Such imperfections could arise, for instance, from innate flaws in the agent, or from such other causes as mutually-conflicting input streams or noisy sensors.

We now prove that K^p is sound and complete for the class of all local-reasoning structures for p frames of mind. The proof rests on first proving two preliminary propositions. We say that a formula is $\mathfrak{L}^p$-*satisfiable* if and only if it is satisfied at some point in a local-reasoning structure for p frames of mind; a formula is K^p-*satisfiable* if and only if it is satisfied at some point in a p-ary relational model.

Proposition 1. For any formula A, if A is $\mathfrak{L}^p$-satisfiable, then A is K^p-satisfiable.

Proof: Suppose some formula A is $\mathfrak{L}^p$-satisfiable. By definition, there exists some local-reasoning structure for p frames of mind, $\mathfrak{L}^p = (S, V, C^p)$, and some point, $s \in S$, such that $(\mathfrak{L}^p, s) \vDash A$. Given such a structure $\mathfrak{L}^p$, we construct a corresponding $(p+1)$-ary relational model $\mathfrak{M}^p$.

Let $\mathfrak{M}^p = (S, V, R^{p+1})$, with S and V just as in $\mathfrak{L}^p$, and relation R^{p+1} defined as follows:

$$(\forall s \in S)(\forall (t_1, \ldots, t_p) \in S^p)\, R^{p+1}(s, t_1, \ldots, t_p) \Leftrightarrow \bigwedge_{i=1}^{p} t_i \in T_i \in C^p(s). \tag{8}$$

It is then easy to show that any formula A is satisfied at any point $s \in S \in \mathfrak{M}^p$ if A is satisfied at that same point in $\mathfrak{L}^p$, by induction on the possible structure of the formula. The case $A = p$, for any propositional variable p, follows directly from the fact that $V \in \mathfrak{M}^p = V \in \mathfrak{L}^p$. The inductive cases for truth-functional connectives are as expected, since the relevant semantic clauses are the same in both frameworks.

For the case $A = \Box B$, for any formula B, since $(\mathfrak{L}^p, s) \vDash \Box B$, by definition of $\mathfrak{L}^p$-satisfaction there exists some frame of mind $T_i \in C^p(s)$ such that $(\forall t \in T_i)\,(\mathfrak{L}^p, t) \vDash B$. Let T_j be such a frame of mind. If $T_j = \emptyset$, then the definition in biconditional (8) yields an empty relation R^{p+1} at s in $\mathfrak{M}^p$, and $(\mathfrak{M}^p, s) \vDash \Box B$ vacuously. If $T_j \neq \emptyset$, then consider an arbitrary p-tuple of points $\bar{t} = (t_1, \ldots, t_p) \in S^p$ such that $R^{p+1}(s, \bar{t})$. Again by the definition in biconditional (8), $t_j \in \bar{t}$ is such that $t_j \in T_j$, and so by assumption $(\mathfrak{L}^p, t_j) \vDash B$. Thus, by the hypothesis of induction, $(\mathfrak{M}^p, t_j) \vDash B$, and since $\bar{t}$ was an arbitrary related p-tuple, we conclude that $(\mathfrak{M}^p, s) \vDash \Box B$. $\Box$

Proving the converse of Proposition 1 is somewhat more complicated. To aid in the proof, we first establish a useful lemma, and then prove a slightly stronger version of the converse, having to do with satisfiable *sets*, Σ, rather than individual formulae, A.

Definition 3. For any set of formulae Σ, the *square-set of* Σ, $[\Sigma]_\Box$, is defined as the set of all formulae taking the $\Box$ operator in Σ. That is: $[\Sigma]_\Box = \{A \mid \Box A \in \Sigma\}$.

Lemma 1. For any set of formulae Σ, any K^p-model $\mathfrak{M}^p = (U, V, R^{p+1})$, and any point $u \in U$, if $(\mathfrak{M}^p, u) \vDash \Sigma$ then either $[\Sigma]_\Box$ is divisible into p subsets, each of which is itself K^p-satisfiable; or R^{p+1} is empty at u.

Proof: Suppose $(\mathfrak{M}^p, u) \vDash \Sigma$, for some set of formulae Σ, K^p-model $\mathfrak{M}^p = (U, V, R^{p+1})$, and point $u \in U$; further, suppose R^{p+1} is not empty at u. That is, there exists some $(v_1, \ldots, v_p) \in U^p$ such that $R^{p+1}(u, v_1, \ldots, v_p)$. Now, either $[\Sigma]_\Box = \emptyset$, or not. On the one hand, if $[\Sigma]_\Box = \emptyset$, then since $\emptyset$ is trivially K^p-satisfiable, $[\Sigma]_\Box$ is divisible into p subsets, $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{p})$, each of which is K^p-satisfiable.

On the other hand, if $[\Sigma]_\Box \neq \emptyset$, then by the semantics for K^p,

$$(\forall A \in [\Sigma]_\Box)(\exists v_i \in (v_1, \ldots, v_p))\,(\mathfrak{M}^p, v_i) \vDash A. \tag{9}$$

Thus, there exists sets $\Gamma_1, \ldots, \Gamma_p$ such that:

1. $(\forall i.1 \leq i \leq p)\,\Gamma_i \subseteq [\Sigma]_\Box$.
2. $(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p) = [\Sigma]_\Box$.
3. $(\forall i.1 \leq i \leq p)\,(\mathfrak{M}^p, v_i) \vDash \Gamma_i$.

That is, each such Γ_i is K^p-satisfiable, and so $[\Sigma]_\Box$ is divisible into p subsets that are K^p-satisfiable. $\Box$

We can now establish the following stronger version of the converse to Proposition 1:

Proposition 2. For any set of formulae Σ, if Σ is K^p-satisfiable, then Σ is $\mathfrak{L}^p$-satisfiable.

The argument is based on the maximum depth of modal nesting in Σ, defined as follows:

Definition 4 (Modal depth). For any formula A, the *depth of* A, *dpt*(A) is defined inductively:

1. For any propositional variable p, *dpt*$(p) = 0$.
2. For negations, *dpt*$(\neg A) =$ *dpt*(A).
3. For conjunctions, *dpt*$(A \wedge B) = \max($*dpt*$(A),$ *dpt*$(B))$
4. For modal formulae, *dpt*$(\Box A) = (1 +$ *dpt*$(A))$.
5. For sets of formulae, *dpt*$(\Sigma) = \max_{A \in \Sigma}$ *dpt*(A).

Proof of Proposition 2: For any set of formulae Σ, our proof is by induction on $n =$ *dpt*(Σ).
Basis ($n = 0$): In this case, Σ is simply a set of purely propositional formulae. Since the K^p-semantics and $\mathfrak{L}^p$-semantics agree over all propositional formulae, if Σ is K^p-satisfiable, it must be $\mathfrak{L}^p$-satisfiable.

For the inductive step, we assume that the hypothesis holds for all values $n < k$, for some $k \in \mathbb{N}$. We now show that it holds for *dpt*$(\Sigma) = k$. In this case, since Σ is K^p-satisfiable, we have that there exists some K^p-model $\mathfrak{M}^p = (U, V, R^{p+1})$ and some point $u \in U$ such that $(\mathfrak{M}^p, u) \vDash \Sigma$. Thus, by Lemma 1, either R^{p+1} is empty at u, or else $[\Sigma]_\Box$ is divisible into p subsets, each of which is K^p-satisfiable. We take the two cases in turn.

I1 (R^{p+1} is empty at u): In this case, define a local-reasoning structure $\mathfrak{L}^p = (\{u\}, V', C^p)$, where:

(a) For any propositional variable p, $V'(p) = \{u\} \Leftrightarrow u \in V(p)$, for $V \in \mathfrak{M}^p$.

(b) $C^p(u) = (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{p})$.

It is then easy to show that for any formula A whatsoever,

$$(\mathfrak{M}^p, u) \vDash A \;\Leftrightarrow\; (\mathfrak{L}^p, u) \vDash A. \tag{10}$$

The proof is by a second induction on the structure of A. The base case for propositional variable p follows directly from (a). The argument for propositional connectives is similarly straightforward, and for formulae of the form $A = \Box B$, we simply note that each such is vacuously satisfied at u in both $\mathfrak{M}^p$ and $\mathfrak{L}^p$, due to the emptiness of relation R^{p+1} and function C^p, respectively. The conditional claim for the set of formulae Σ then follows *a fortiori*.

I2 ($[\Sigma]_\Box$ is divisible into p K^p-satisfiable subsets): In this case, there exists sets $\Gamma_1, \ldots, \Gamma_p$ such that:

1. $(\forall i.1 \leq i \leq p)\, \Gamma_i \subseteq [\Sigma]_\Box$.
2. $(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p) = [\Sigma]_\Box$.
3. $(\forall i.1 \leq i \leq p)\, \Gamma_i$ is K^p-satisfiable.

Now, for each Γ_i, since each $A \in \Gamma_i$ is such that $\Box A \in \Sigma$, $\mathit{dpt}(\Gamma_i) < \mathit{dpt}(\Sigma) = k$. Therefore, by the hypothesis of the induction, each such Γ_i is $\mathfrak{L}^p$-satisfiable; that is:

$$\begin{aligned} &(\exists \mathfrak{L}^p_1 = (S_1,\, V_1,\, C^p_1))(\exists s^\star_1 \in S_1)\, (\mathfrak{L}^p_1, s^\star_1) \vDash \Gamma_1 \\ \wedge\, &(\exists \mathfrak{L}^p_2 = (S_2,\, V_2,\, C^p_2))(\exists s^\star_2 \in S_2)\, (\mathfrak{L}^p_2, s^\star_2) \vDash \Gamma_2 \\ &\qquad\qquad\vdots \\ \wedge\, &(\exists \mathfrak{L}^p_p = (S_p,\, V_p,\, C^p_p))(\exists s^\star_p \in S_p)\, (\mathfrak{L}^p_p, s^\star_p) \vDash \Gamma_p \end{aligned} \tag{11}$$

Without loss of generality, we can assume that for any Γ_i, Γ_j ($i \neq j$), $(\Gamma_i \cap \Gamma_j) = \emptyset$ (i.e., each set of points is distinct), and furthermore that for all Γ_i, the point $u \in U \in \mathfrak{M}^p$ is such that $u \notin \Gamma_i$. We can then define a local-reasoning structure $\mathfrak{L}^p_+ = (S_+,\, V_+,\, C^p_+)$ as follows:

(a) $S_+ = (S_1 \cup S_2 \cup \cdots \cup S_p \cup \{u\})$.

(b) For any propositional variable p,

$$V_+(p) = \begin{cases} V_1(p) \cup V_2(p) \cup \cdots \cup V_p(p) \cup \{u\} & \text{if } u \in V(p), \text{ for } V \in \mathfrak{M}^p; \\ V_1(p) \cup V_2(p) \cup \cdots \cup V_p(p) & \text{else.} \end{cases}$$

(c) For any point $s \in S_+$,

$$C^p_+(s) = \begin{cases} C_i(s) & \text{if } s \in S_i \ (1 \leq i \leq p); \\ (\{s^\star_1\},\, \{s^\star_2\}, \ldots,\, \{s^\star_p\}) & \text{if } s = u. \end{cases}$$

It then follows that for any formula $A \in \Sigma$, if $(\mathfrak{M}^p, u) \vDash A$ then $(\mathfrak{L}^p_+, u) \vDash A$. We first show that for any value of i ($1 \leq i \leq p$), and for any formula B whatsoever,

$$(\forall s \in S_i)\, (\mathfrak{L}^p_i, s) \vDash B \;\Leftrightarrow\; (\mathfrak{L}^p_+, s) \vDash B. \tag{12}$$

This proof is by a secondary induction on the structure of B, and follows immediately in every case from clauses (b) and (c) above. This proof is then followed by a third induction on the structure of $A \in \Sigma$. The basis case, for propositional variable p, follows immediately from the definition of V_+ in (b); the inductive case for propositional connectives is also as expected. For modal formula $A = \Box C$, we note that $C \in [\Sigma]_\Box$, and so there exists some $\Gamma_i \subseteq [\Sigma]_\Box$ such that $C \in \Gamma_i$. Therefore, we have that:

$$(\mathfrak{L}^p_1, s^\star_1) \vDash C \;\vee\; (\mathfrak{L}^p_2, s^\star_2) \vDash C \;\vee \cdots \vee\; (\mathfrak{L}^p_p, s^\star_p) \vDash C. \tag{13}$$

So suppose $(\mathfrak{L}^p_j, s^\star_j) \vDash C$. Then, by the previous step in this case, $(\mathfrak{L}^p_+, s^\star_j) \vDash C$, and so by the construction in (c),

$$(\exists T_j \in C^p_+(u))(\forall s \in T_j)\,(\mathfrak{L}^p_+, s) \vDash C, \tag{14}$$

which means that $(\mathfrak{L}^p_+, u) \vDash \Box C = A$, as required. □

With these facts in hand, the following new theorems hold:

Theorem 1 (Equivalence of Validity). Any formula A is K^p-valid if and only if A is $\mathfrak{L}^p$-valid.

Proof: We prove the contrapositive of each direction of the biconditional. For the left-to-right direction, assume that A is not $\mathfrak{L}^p$-valid. That is, $(\exists \mathfrak{L}^p = (S, , V, C^p))(\exists s \in S)\,(\mathfrak{L}^p, s) \nvDash A$. Thus, by the definition of the semantics for local-reasoning structures, $(\mathfrak{L}^p, s) \vDash \neg A$, and so $\neg A$ is $\mathfrak{L}^p$-satisfiable. Therefore, by Proposition 1, $\neg A$ is K^p-satisfiable; that is to say, $(\exists \mathfrak{M}^p = (U, V', R^{p+1}))(\exists u \in U)\,(\mathfrak{M}^p, u) \vDash \neg A$. So, by the semantics for K^p, $(\mathfrak{M}^p, u) \nvDash A$, and therefore A is not K^p-valid.

For the right-to-left direction, we assume that A is not K^p-valid. The argument that A is therefore not $\mathfrak{L}^p$-valid uses Proposition 2, and is essentially identical in form to the one just given, so long as we note the obvious fact that a formula A is $\mathfrak{L}^p$-satisfiable (K^p-satisfiable) if and only if the singleton set $\{A\}$ is $\mathfrak{L}^p$-satisfiable (K^p-satisfiable). □

Theorem 2 (Characterization). $(\forall p \in \mathbb{N})$ K^p is sound and complete with respect to the class of all local-reasoning structures for p frames of mind.

Proof: The claim follows directly from the soundness and completeness of each K^p with respect to its associated class of models. For soundness with respect to local-reasoning structures for p frames of mind, we note that if formula A is a K^p-theorem, then by the existing soundness result, A is K^p-valid. From Theorem 1, it follows that A is $\mathfrak{L}^p$-valid. For completeness with respect to local-reasoning structures for p frames of mind, we note that if A is $\mathfrak{L}^p$-valid, then by Theorem 1, A is K^p-valid. Thus, by the existing completeness result, A is a K^p-theorem. □

The logic N is known to be sound and complete for local-reasoning structures with arbitrarily many frames of mind [5, §9.6]. Theorem 2 entails that this claim is in fact equivalent to the known result that N is the intersection of all the logics in the K^p hierarchy [9]. This shows an interesting connection between the K^p hierarchy and work on logical omniscience and inconsistent belief. These logics model productive reasoning for agents with beliefs of some fixed level of possible inconsistency (resulting, e.g. , from a fixed set of internally consistent, but conflicting input-streams). Stronger than N, they can represent epistemic agents with more powerful reasoning strategies than previously considered.

3 Complexity Results

Ladner [12] shows that the satisfiability question for any uni-modal logic S such that $K \leq S \leq S4$ is PSPACE-complete. Halpern and Moses [7] extend this result to systems using multiple knowledge-operators for multiple agents. We follow their lead, extending the result to the entire K^p-hierarchy and showing that $(\forall p \in \mathbb{Z}^+)$ K^p-SAT is PSPACE-complete.

3.1 Lower Bounds

To prove PSPACE-hardness, we also reduce from the problem of validity for Quantified Boolean Formulae (QBF-VALID), which is PSPACE-complete [17]. For any QBF, $A \equiv Q_1 p_1 Q_2 p_2 \ldots Q_m p_m A'$ (where each $Q_i \in \{\forall, \exists\}$ and A' is quantifier-free, containing only propositional letters $p_1, \ldots, p_m$), and any logic K^p $(p > 1)$, we derive a modal formula B_{K^p} that is K^p-satisfiable if and only if A is QBF-valid. Intuitively, for QBF A with m quantifiers, B_{K^p} is such that modal structures satisfying it mimic a tree of height m, with leaves comprising the set of propositional truth-assignments that demonstrate A's validity. We proceed by defining several important subformulae.

First, a *depth* formula, giving the propositions d_i satisfied at each depth in the tree:

$$[\text{dpt}] \quad \equiv \quad \bigwedge_{i=1}^{m+1} (d_i \to d_{i-1}). \tag{15}$$

Next, a *branching* formula, which forces the appropriate branch-structure onto the model. Simply put, whenever quantifier Q_{i+1} in A is $\forall$, nodes at depth i in the tree will have two successors at the next depth $(i+1)$, one of which makes p_{i+1} true, and the other of which makes it false. If $Q_{i+1} = \exists$, nodes at depth i will have a single successor, where p_{i+1} gets the value appropriate to satisfy the overall QBF. For readability, we use the dual modal operator $\Diamond \equiv \neg\Box\neg$, reading $\Diamond B$ as "there exists some successor at which B holds".

$$\begin{aligned}[\text{brg}] \equiv & \bigwedge_{\{i \,|\, Q_{i+1} = \forall\}} \big((d_i \wedge \neg d_{i+1}) \to \big[\Diamond(d_{i+1} \wedge \neg d_{i+2} \wedge p_{i+1}) \wedge \Diamond(d_{i+1} \wedge \neg d_{i+2} \wedge \neg p_{i+1})\big]\big) \\ & \qquad\qquad \wedge \\ & \bigwedge_{\{i \,|\, Q_{i+1} = \exists\}} \big((d_i \wedge \neg d_{i+1}) \to \Diamond(d_{i+1} \wedge \neg d_{i+2})\big).\end{aligned} \tag{16}$$

So far, this is just as in the existing proofs. For any logic K^p $(p > 1)$, however, we must contend with complications arising from the lack of conjunctive closure for $\Box$. Recall that in K^2, for example, the fact that two formulae $\Box B$ and $\Box B'$ are both true at some point x does not mean that the conjunction $(B \wedge B')$ is true at any *single* related point y in the model; since models for K^2 employ a 3-ary relation between points x and pairs of points (y_1, y_2), all that is necessary is that each of B, B' be true at *one* y_i in each such pair, not that there be some y_i where *both* are true. For the rest of the proof to carry through, however, we will need to find some way to ensure that this is indeed the case.

To solve this problem, we define a *path-forcing* formula, $[\text{path}_p]$, inductively for values $p > 1$:

$$\begin{aligned}[\text{path}_2] &\equiv \Box\neg f_1 \\ [\text{path}_3] &\equiv \Box(\neg f_1 \wedge f_2) \wedge \Box(\neg f_1 \wedge \neg f_2) \\ [\text{path}_4] &\equiv \Box(\neg f_1 \wedge f_2) \wedge \Box(\neg f_1 \wedge \neg f_2 \wedge f_3) \wedge \Box(\neg f_1 \wedge \neg f_2 \wedge \neg f_3) \\ &\ \vdots \\ [\text{path}_p] &\equiv \Box(\neg f_1 \wedge f_2) \wedge \Box(\neg f_1 \wedge \neg f_2 \wedge f_3) \wedge \cdots \\ &\qquad \wedge \Box(\neg f_1 \wedge \neg f_2 \wedge \cdots \wedge \neg f_{p-2} \wedge f_{p-1}) \wedge \Box(\neg f_1 \wedge \neg f_2 \wedge \cdots \wedge \neg f_{p-2} \wedge \neg f_{p-1})\end{aligned} \tag{17}$$

It is easy to see that $[\text{path}_p]$ allows us to ensure that a pair of formulae are both satisfied at some one particular point in any related p-tuple, as desired. For any B and B', and any logic K^p $(p > 1)$, if the formula

$$\Box(B \wedge f_1) \wedge \Box(B' \wedge f_1) \wedge [\text{path}_p] \tag{18}$$

is satisfied at a point x in some model for K^p, then any related p-tuple $(y_1, \ldots, y_p)$ must contain at least one y_i satisfying $(B \wedge B')$, since the remaining points y_j in any such tuple must all satisfy $\neg f_1$.

To complete our reduction, we define a *determining* formula, $[\text{dtd}^\star]$, which ensures that for any point x at some depth i in our tree, whatever value p_i gets at x is "passed down" to all points on branches below x:

$$[\text{dtd}^\star] \;\equiv\; \bigwedge_{i=1}^{m} \big(d_i \rightarrow \big[\big(p_i \rightarrow \Box((d_i \rightarrow p_i) \wedge f_1)\big) \wedge \big(\neg p_i \rightarrow \Box((d_i \rightarrow \neg p_i) \wedge f_1)\big)\big]\big). \tag{19}$$

We also introduce a notational convention: for any formula B, formula $\Box^i_{f_1} B$ is defined inductively on i:

$$\begin{aligned} \Box^0_{f_1} B &\equiv B \\ \Box^1_{f_1} B &\equiv \Box(B \wedge f_1) \\ \Box^2_{f_1} B &\equiv \Box(\Box(B \wedge f_1) \wedge f_1) \\ &\;\vdots \\ \Box^i_{f_1} B &\equiv \Box(\Box^{i-1}_{f_1} B \wedge f_1) \end{aligned} \tag{20}$$

Finally, for any logic K^p ($p > 1$), and for any QBF $A = Q_1 p_1 Q_2 p_2 \ldots Q_m p_m A'$, define B_{K^p}:

$$d_0 \wedge \neg d_1 \wedge \bigwedge_{i=0}^{m} \Box^i_{f_1} \big([\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_m \rightarrow A') \wedge [\text{path}_p]\big). \tag{21}$$

As an example, consider QBF $A \equiv \forall p_1 \exists p_2 (A')$, and logic K^2, with a 3-ary accessibility relation. Here, the corresponding formula B_{K^2} is:

$$\begin{aligned} &d_0 \wedge \neg d_1 \wedge [\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow A') \wedge \Box\neg f_1 \\ &\quad \wedge \Box\big(([\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow A') \wedge \Box\neg f_1) \wedge f_1\big) \\ &\quad \wedge \Box\big(\Box\big(([\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow A') \wedge \Box\neg f_1) \wedge f_1\big) \wedge f_1\big) \end{aligned} \tag{22}$$

and we have the following component formulae:

$$[\text{dpt}] \equiv (d_1 \rightarrow d_0) \wedge (d_2 \rightarrow d_1) \wedge (d_3 \rightarrow d_2) \tag{23}$$

$$\begin{aligned} [\text{brg}] \equiv\; &\big((d_0 \wedge \neg d_1) \rightarrow [\Diamond(d_1 \wedge \neg d_2 \wedge p_1) \wedge \Diamond(d_1 \wedge \neg d_2 \wedge \neg p_1)]\big) \\ \wedge &\big((d_1 \wedge \neg d_2) \rightarrow \Diamond(d_2 \wedge \neg d_3)\big) \end{aligned} \tag{24}$$

$$\begin{aligned} [\text{dtd}^\star] \equiv\; &\big(d_1 \rightarrow \big[(p_1 \rightarrow \Box((d_1 \rightarrow p_1) \wedge f_1)) \wedge (\neg p_1 \rightarrow \Box((d_1 \rightarrow \neg p_1) \wedge f_1))\big]\big) \\ \wedge &\big(d_2 \rightarrow \big[(p_2 \rightarrow \Box((d_2 \rightarrow p_2) \wedge f_1)) \wedge (\neg p_2 \rightarrow \Box((d_2 \rightarrow \neg p_2) \wedge f_1))\big]\big) \end{aligned} \tag{25}$$

In general, the number of conjuncts in B_{K^p}, and in all its subformulae, is directly proportional to m, the number of quantifiers in the QBF A. The size of $[\text{path}_p]$ is proportional to p. Thus, for a given K^p the overall size of B_{K^p} will be polynomial in $|A| = n$, for sufficiently large n.

3.1.1 Proof of Equivalence under Reduction

We are now in position to prove the following proposition:

Proposition 3. $(\forall p > 1)(\forall A)$ A is QBF-VALID if and only if $B_{K^p} \in K^p$-SAT.

Proof: We give only one direction of the proof. For the proof of the right-to-left direction, we rely upon a natural induction on the number of quantifiers m in A to show the required result. The full details can be found in [1]. We prove the left-to-right direction in full. For arbitrary value $p > 1$ and arbitrary formula A, assume $A \in$ QBF-VALID. We then use the technique of Ladner [12] to construct a K^p model, $\mathfrak{M}_A^p = (U_A,\, V_A,\, R_A^{p+1})$, such that $(\exists u \in U_A)\,(\mathfrak{M}_A^p,\, u) \vDash B_{K^p}$. The construction and proof proceed as follows:

I. Let U_A be a finite subset of the numerical strings $\{0,\, 1,\, \ldots,\, (2p-1)\}^\star$, defined inductively by:

1. The empty string $\lambda \in U_A$.
2. For any u, if $u \in U_A$ & $|u| = i < m$, then:
   (a) The $2p$ distinct strings, $u0,\, u1,\, \ldots,\, u(2p-1)$ are all in U_A, if $Q_{i+1} = \forall$.
   (b) The p distinct strings, $u0,\, u1,\, \ldots,\, u(p-1)$ are all in U_A, if $Q_{i+1} = \exists$.
3. No other strings are in U_A.

II. Define the accessibility relation R_A^{p+1} (hereafter, simply R_A) such that

$$(\forall u \in U_A)(\forall (v_1, \ldots, v_p) \in (U_A)^p)\, R_A(x,\, v_1, \ldots, v_p)$$

if and only if both of the following hold:

1. $(v_1 = u0)$ or $(v_1 = up)$.
2. $(\forall v_i,\, 1 \leq i < p)\,(v_i = un) \Leftrightarrow (v_{i+1} = u(n+1))$.

That is, our model will consist of a tree, with nodes consisting of strings of symbols. Each string at depth i of the tree will have length equal to i, and will have two p-tuples of children if quantifier $Q_{i+1} = \forall$ and one p-tuple of children if $Q_{i+1} = \exists$. For any string u, and any tuple $(v_1, v_2,\, \ldots,\, v_p)$ of its children, u will be a prefix of each child-string v_i, and each tuple will consist of p consecutive strings occurring in order, beginning either with $u0$ or up.

III. To simplify notation, we employ the following convention from now on, extending valuation function $V_A : Prop \to 2^U$ to the function $V_A : (Prop \times U) \to \{$TRUE, FALSE$\}$ defined as follows:

$$V_A(p,\, u) = \begin{cases} \text{TRUE} & \text{if } u \in V_A(u); \\ \text{FALSE} & \text{else.} \end{cases} \tag{26}$$

We now want to define valuation function V_A so that the following holds for any $u \in U_A$:

(C1) If $|u| = i$, then $V_A(d_i,\, u) =$ TRUE.

(C2) If $|u| = |u'| = i$, and some v is such that both $R_A(v, u_1, u_2,\, \ldots,\, u_p)$ & $R_A(v, u'_1, u'_2,\, \ldots,\, u'_p)$ and also $u \in (u_1, u_2,\, \ldots,\, u_p)$ & $u' \in (u'_1, u'_2,\, \ldots,\, u'_p)$, then the following all hold:

   (a) $(\forall u_j,\, u_k)\; V_A(p_{i+1}, u_j) = V_A(p_{i+1}, u_k)$.

(b) $(\forall u'_j,\, u'_k)\; V_A(p_{i+1}, u'_j) = V_A(p_{i+1}, u'_k)$.

(c) $(\forall u_j,\, u'_k)\; V_A(p_{i+1}, u_j) \neq V_A(p_{i+1}, u'_k)$.

(C3) If $|u| = i > j$, then $V_A(p_j, u) = V_A(p_j, u')$, where u' is a prefix of u (i.e. $u = u'x$), and $|u'| = (i-1)$.

(C4) If $u = u'x$, with $0 \leq x \leq (2p-1)$, then for any f_j, $V_A(f_j, u) =$ TRUE if and only if suffix $x = (j-1)$ or $x = p + (j-1)$.

(C5) If $|u| = i$, then it must be true that

$$Q_{i+1}p_{i+1} \dots Q_m p_m A'[V_A(p_1, u),\, \dots,\, V_A(p_i, u)] \in \text{QBF-VALID},$$

where $A'[V_A(p_1, u),\, \dots,\, V_A(p_i, u)]$ is simply the formula A', with every occurrence of any variable p_j replaced by the corresponding value $V_A(p_j, u) \in \{$TRUE, FALSE$\}$.

IV. We show that we can create the appropriate V_A as follows, inductively on $i = |u|$. For any value j, set $V_A(d_j, u) =$ TRUE if $j \leq i = |u|$, and FALSE otherwise. Further, for any value k, set:

$$V_A(p_k, u) = \begin{cases} \text{FALSE} & \text{if } k > i = |u|\,. \\ V_A(p_k, u') & \text{if } (k < i = |u|) \;\&\; (u = u'x) \;\&\; (|u'| = i-1). \end{cases} \tag{27}$$

(Note that we have not yet set $V_A(p_k, u)$ for the case $k = i$.) Lastly, for any f_j, set $V_A(f_j, u) =$ TRUE (i.e., $u \in V_A(f_j)$) if and only if $u = u'x$, for some string u' and some x, and either $x = (j-1)$ or $x = p+(j-1)$.

Now, if $i = |u| = 0$, then it is easily seen that desired properties (C1)–(C4) described above in **III** all hold by the definition of V_A just given. Furthermore, for (C5), the formula

$$Q_{i+1}p_{i+1} \dots Q_m p_m A'[V_A(p_1, u),\, \dots,\, V_A(p_i, u)]$$

is simply the original QBF, $A \equiv Q_1 p_1 \dots Q_m p_m A'$, and so is in QBF-VALID by hypothesis.

For the inductive step of our proof, we assume that conditions (C1)–(C5) hold for any $j < i = |u|$, and show how to set $V_A(p_k, u)$ for the remaining case $k = i$. If $Q_i = \forall$, set:

$$V_A(p_i, u) = \begin{cases} \text{TRUE} & \text{if the last digit in } u \text{ is } x,\ 0 \leq x \leq (p-1). \\ \text{FALSE} & \text{else.} \end{cases} \tag{28}$$

That is, p_i holds at *exactly one* of the related p tuples of points. If $Q_i = \exists$, then by the inductive hypothesis, there must be some $V \in \{$TRUE, FALSE$\}$ such that:

$$Q_1 p_1 Q_2 p_2 \dots Q_m p_m A'[V_A(p_1, u),\, \dots,\, V_A(p_{i-1}, u), V] \in \text{QBF-VALID}, \tag{29}$$

and so we set $V_A(p_i, u) = V$. It thus follows that conditions (C1)–(C5) hold for $|u| = i > j$, and thus for any i.

Finally, it is straightforward, given the definitions above, that $(\mathfrak{M}^p, \lambda) \vDash B_{K^p}$, and so that $B_{K^p} \in K^p$-SAT. Therefore, since value p and QBF A were arbitrary, we have proven one direction of Proposition 3, namely $(\forall p > 1)(\forall A)(A \in \text{QBF-VALID} \Rightarrow B_{K^p} \in K^p\text{-SAT})$. $\square$

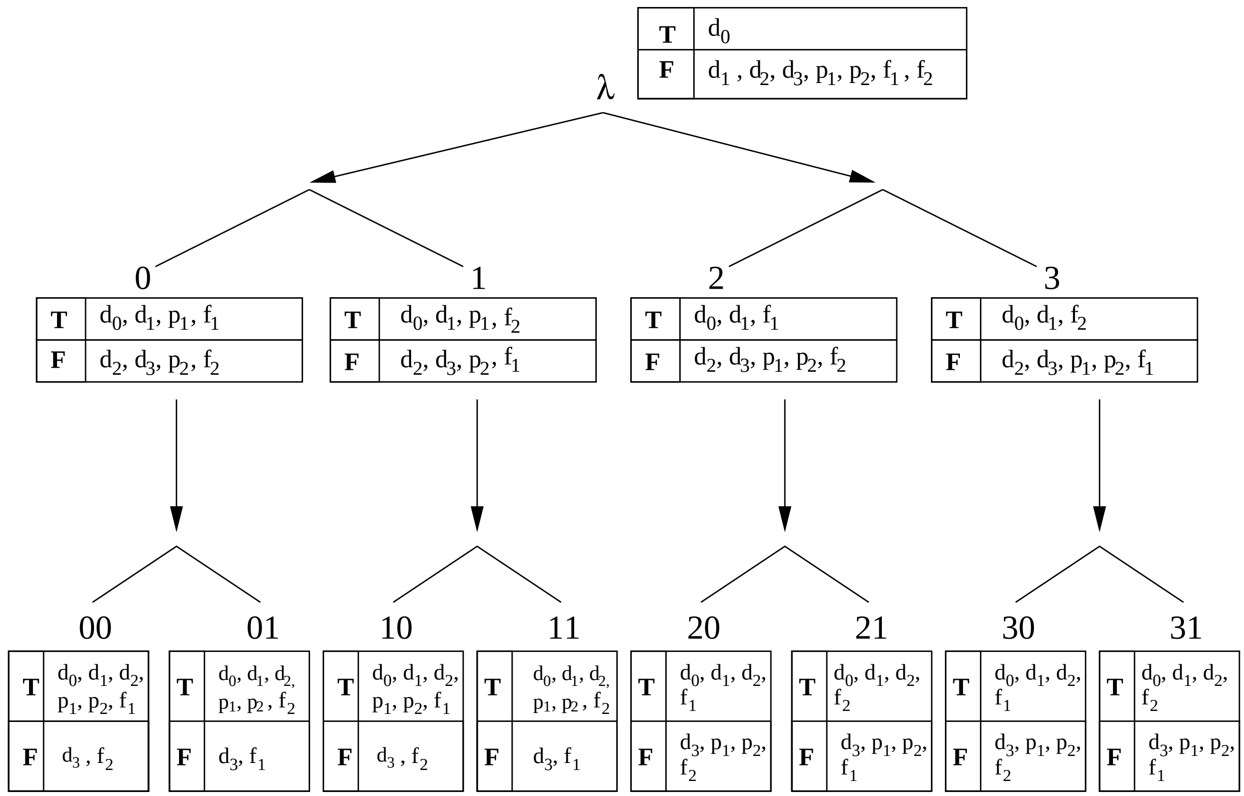

Figure 2: Model for QBF $\forall p_1 \exists p_2 (p_1 \equiv p_2)$, constructed for logic K^2. At each point, the set of sentences that are true (**T**) or false (**F**) are identified.

3.1.2 A Sample Model $\mathfrak{M}^p_A$

Figure 2 shows an example of the model $\mathfrak{M}^p_A$ generated according to the definition given in the preceding proof. The QBF in this case is $A \equiv \forall p_1 \exists p_2 (p_1 \equiv p_2)$, and the logic is K^2, in which we have a ternary accessibility relation R^3_A. It is easy to check that $(\mathfrak{M}^p_A, \lambda) \models B_{K^2}$, where B_{K^2} is:

$$\begin{aligned} d_0 \wedge \neg d_1 \wedge [\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow (p_1 \equiv p_2)) \wedge \Box\neg f_1 \\ \wedge \Box\big(([\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow (p_1 \equiv p_2)) \wedge \Box\neg f_1) \wedge f_1\big) \\ \wedge \Box\big(\Box\big(([\text{dpt}] \wedge [\text{brg}] \wedge [\text{dtd}^\star] \wedge (d_2 \rightarrow (p_1 \equiv p_2)) \wedge \Box\neg f_1) \wedge f_1\big) \wedge f_1\big), \end{aligned} \tag{30}$$

and we have the following component formulae:

$$[\text{dpt}] \equiv (d_1 \rightarrow d_0) \wedge (d_2 \rightarrow d_1) \wedge (d_3 \rightarrow d_2); \tag{31}$$

$$\begin{aligned} [\text{brg}] \equiv & \big((d_0 \wedge \neg d_1) \rightarrow [\Diamond(d_1 \wedge \neg d_2 \wedge p_1) \wedge \Diamond(d_1 \wedge \neg d_2 \wedge \neg p_1)]\big) \wedge \\ & \big((d_1 \wedge \neg d_2) \rightarrow \Diamond(d_2 \wedge \neg d_3)\big); \end{aligned} \tag{32}$$

$$\begin{aligned} [\text{dtd}^\star] \equiv & \big(d_1 \rightarrow \big[(p_1 \rightarrow \Box((d_1 \rightarrow p_1) \wedge f_1)) \wedge (\neg p_1 \rightarrow \Box((d_1 \rightarrow \neg p_1) \wedge f_1))\big]\big) \wedge \\ & \big(d_2 \rightarrow \big[(p_2 \rightarrow \Box((d_2 \rightarrow p_2) \wedge f_1)) \wedge (\neg p_2 \rightarrow \Box((d_2 \rightarrow \neg p_2) \wedge f_1))\big]\big). \end{aligned} \tag{33}$$

3.1.3 Proof of PSPACE-Hardness

Given the equivalence result of the preceding section, the following theorem, establishing the lower-bound result we wanted, is immediate given the known hardness of QBF-VALID:

Theorem 3. For any $(p > 1)$, K^p-SAT is PSPACE-hard.

Here, we note that this result means that the conjecture of [21] is in fact untrue. Each logic K^p $(p > 1)$ is strictly weaker than $K = K^1$; none of these logics, in particular, has the full conjunctive closure axiom $[K]$, and yet satisfiability for each is indeed a PSPACE-hard problem. Furthermore, it is a result of [9] that logic N is the intersection of the K^p logics: $N = \bigcap_{p \in \mathbb{N}} K^p$, and so PSPACE-hardness does not extend below that hierarchy. However, the conjecture is very nearly right; what is necessary for the jump from NP (in the case of N) to PSPACE (in the case of any K^p) is *some degree* of conjunctive closure, given by some version, howsoever weak, of axiom $[K^p]$.

3.2 Upper Bounds

To establish K^p-SAT $\in$ PSPACE we employ *modal tableaux*, first introduced by Kripke [11] as an extension of the familiar notion of a propositional tableau. In essence, a propositional tableau Δ is a set of formulae, closed under elementary propositional equivalence. Given that definition, we can then define modal tableaux, based upon the propositional version. In both definitions, Φ is the set of all formulae.

Definition 5 (Propositional tableau). The set $\Delta \subseteq \Phi$ is a *propositional tableau* if and only if:

1. $(\forall B \in \Phi)\ \neg\neg B \in \Delta \Rightarrow B \in \Delta$.
2. $(\forall B \in \Phi)\ B \in \Delta \Rightarrow \neg B \notin \Delta$ and $\neg B \in \Delta \Rightarrow B \notin \Delta$.
3. $(\forall B, B' \in \Phi)\ B \wedge B' \in \Delta \Rightarrow B \in \Delta$ and $B' \in \Delta$.
4. $(\forall B, B' \in \Phi)\ \neg(B \wedge B') \in \Delta \Rightarrow \neg B \in \Delta$ or $\neg B' \in \Delta$.

Definition 6 (K^p-tableau). For any K^p, a K^p*-tableau* is a structure $\mathfrak{T} = (W, R^{p+1}, L)$ such that: (1) W is a set of points. (2) $R^{p+1} \subseteq W^{p+1}$ is a $(p+1)$-ary accessibility relation. (3) $L : W \rightarrow 2^{\Phi}$ is a *labelling function*, identifying each point in W with a subset of Φ; for any $w \in W$:

1. $L(w)$ is a propositional tableau.
2. If $\Box B \in L(w)$ and $\exists(v_1, \ldots, v_p)\, R(w, v_1, \ldots, v_p)$, then $\exists v_i \in (v_1, \ldots, v_p)$ such that $B \in L(v_i)$.
3. If $\neg\Box B \in L(w)$, then $\exists(v_1, \ldots, v_p)\, R(w, v_1, \ldots, v_p)$ and $\forall v_i \in (v_1, \ldots, v_p)\ \neg B \in L(v_i)$.

For any $B \in \Phi$ and any K^p-tableau $\mathfrak{T} = (W, R^{p+1}, L)$, we say that $\mathfrak{T}$ *is a tableau for* B if and only if there exists some point $w \in W$ such that $A \in L(w)$.

A K^p-tableau is essentially a $(p+1)$-ary modal frame whose points are propositional tableaux. In fact, we show that K^p-tableaux can serve as proper K^p-models, proving:

Theorem 4. $(\forall B \in \Phi)(\forall p \in \mathbb{Z}^+)\ B \in K^p$-SAT if and only if some K^p-tableau $\mathfrak{T}$ is a tableau for B.

Proof (Sketch): The left-to-right direction is straightforward. For arbitrary formula B and arbitrary value $p \in \mathbb{Z}^+$, suppose that $B \in K^p$-SAT. Then there exists a K^p-model $\mathfrak{M} = (W, R^{p+1}, V)$ such that $(\exists w \in W)\ (\mathfrak{M}, w) \models B$. It is then straightforward to construct a tableau $\mathfrak{T}$ for A on the basis of the model $\mathfrak{M}$.

For the other direction of proof, assume that for arbitrary formula B and arbitrary value $p \in \mathbb{Z}^+$, there exists some K^p-tableau $\mathfrak{T} = (W, R^{p+1}, L)$ such that $\mathfrak{T}$ is a tableau for B. Let $At \subseteq \Phi$ be the set of atomic propositional formulae. Then define a model $\mathfrak{M} = (W, R^{p+1}, V)$ with W and R^{p+1} just as in $\mathfrak{T}$, and set V as follows:

$$(\forall a \in At)(\forall w \in W)\ V(a, w) = \mathrm{T} \ \Leftrightarrow\ p \in L(w). \tag{34}$$

We can now show that for any point $w \in W$ such that $B \in L(w)$ and any B', if B' is a subformula of B, we have both that $(B' \in L(w) \Rightarrow (\mathfrak{M}, w) \models B')$, and also $(\neg B' \in L(w) \Rightarrow (\mathfrak{M}, w) \not\models B')$. This suffices for the necessary result. The proof is by induction on n, the number of connectives occurring in B'. $\square$

3.2.1 Constructing and Checking K^p-Tableaux

The mere existence of K^p-tableaux for every satisfiable formula is clearly not enough to establish our proof. We require a method for actually constructing such a tableau, given formula B and logic K^p, and checking satisfiability on it, using only polynomial space. In aid of this, we provide an algorithm that takes formula B and value p as input and constructs a "pre-tableau," T. This pre-tableau takes the form of a $(p+1)$-ary tree in which each node t is labelled with $L(t) = (\Sigma_t, i)$, where Σ_t is a set of formulae, and $i \in \{0, 1\}$ indicates whether that set is satisfiable (1) or not (0). Although the full details of our algorithm cannot be presented here for reasons of space, the procedure works by essentially (1) building a propositional tableau Δ using subformulae of B, and (2) adding $(p+1)$-ary branches for every subformula of the form $\neg\Box B'$ that we encounter. It is easy to prove that the algorithm given is guaranteed to terminate, since the tree T it produces is of height polynomial in the size of formula B, and show:

Theorem 5. $(\forall B)(\forall p \in \mathbb{Z}^+)\ B \in K^p$-SAT $\Leftrightarrow$ the K^p-tableau construction algorithm returns a tree T with root t_0 and label $L(t_0) = (\{B\}, 1)$.

Now, it is a known fact from traditional modal logic that some formulae B can *only* be satisfied in structures of size exponential in $|B|$. Indeed, the tree-structure T produced by our algorithm can be very large—although its overall height is polynomially bounded, the branching factor can make for a great number of nodes. Thus, we must show that T can be constructed and checked in a depth-first manner, one branch at a time, re-using space as we go, so that we need only store data for that single branch at each pass.

Here, the challenge comes in the step for constructing and checking the $(p+1)$-ary branches, since we must check each element of every possible p-partition of $\Box[\Sigma_t] = \{B' \mid \Box B' \in \Sigma_t\}$. Figure 3 shows an example for K^2: since Σ_t contains $\neg\Box D$, the tableau construction requires the generation of pairs of successors, with $\neg D$ true at each. These successors, in turn, comprise all ways of dividing up the set of sentences in $\Box[\Sigma_t] = \{A, B, C\}$ into two parts, reflecting all the ways in which it is possible to satisfy the semantic clause for the $\Box$ operator at point t.

Since the size of $\Box[\Sigma_t]$ is bounded only by $|B| = n$, the set of all p-partitions can have size exponential in n, and thus we cannot keep track of all corresponding branches explicitly, as in existing proofs for other logics. Instead, we adapt a combinatorial algorithm of Even [4], generating partitions individually using $(4 \cdot (|\Box[\Sigma_t]| + 1) \cdot \log p)$ bits, re-using space as necessary. Since $|\Box[\Sigma_t]| \leq n$, generating partitions uses no more than $(4n \log p + 4)$ bits. An additional $(5n + p)$ bits suffices to do all necessary record-keeping; so, since the height of our tree is bounded by n, we need no more than $N \leq (5 + 4\log p)n^3 + pn^2$ bits to compute K^p-SAT, which is $\mathcal{O}(n^3) = \mathcal{O}(|B|^3)$, for sufficiently large formulae B, establishing:

Theorem 6. For any value $p \in \mathbb{Z}^+$ and formula B, K^p-SAT is solvable in space polynomial in size $|B| = n$.

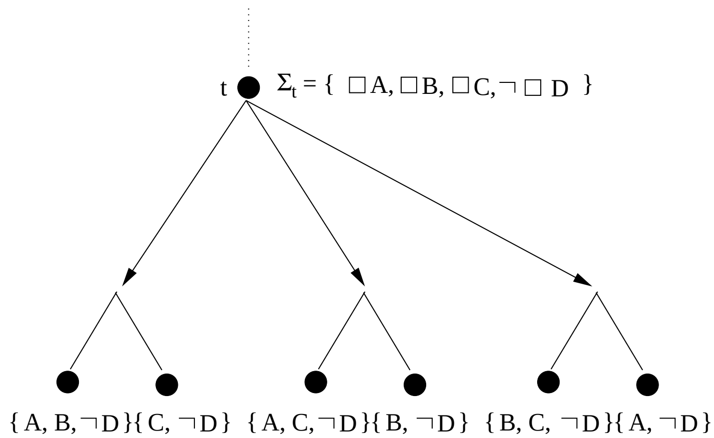

Figure 3: A simple tableau example, for logic K^2.

This result, then, combined with Theorem 3, establishes the main result, namely:

Theorem 7. $(\forall p \in \mathbb{Z}^+)$ K^p-SAT is PSPACE-complete.

4 Conclusions and Extensions

We have shown a hierarchy of weak modal logics between K and N to be sound and complete for epistemic models of agents with multiple "frames of mind." These logics are stronger than N, providing potentially interesting tools for modelling reasoning with inconsistent information. Our ongoing research investigates the use of such logics for reasoning about distributed systems and security protocols, and the related model-checking problem, as in [5, 18, 19].

Our complexity results extend existing work [12, 7] and provide an answer to a conjecture of Vardi [21]. We have shown that the precise boundary between NP and PSPACE with respect to modal logics of knowledge is marked by the presence of conjunctive closure principles—although not, as originally hypothesized, the strong closure axiom $[K]$. This shows not only the difficulty of reasoning with inconsistent information, but also isolates relations between computational complexity and an important structural features of common modal logics.

Our PSPACE-completeness result holds for all logics K^p, and is shown for a single modality; it should be straightforward, and may be interesting, to extend this result to multiple knowledge-operators, for use in reasoning about multiple epistemic agents. Further results, including questions of model-checking for K^p logics, and the NP-completeness of K^p-SAT when the depth of nesting for modal operators is bounded by some fixed and finite amount (in the spirit of Halpern [6]) can be found in [14, 1].

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