Interactive Unawareness Revisited

Joseph Y. Halpern
Computer Science Department
Cornell University, U.S.A.
e-mail: halpern@cs.cornell.edu

Leandro Chaves Rêgo
School of Electrical and Computer Engineering
Cornell University, U.S.A.
e-mail: lcr26@cornell.edu

Abstract

We analyze a model of interactive unawareness introduced by Heifetz, Meier and Schipper (HMS). We consider two axiomatizations for their model, which capture different notions of validity. These axiomatizations allow us to compare the HMS approach to both the standard (S5) epistemic logic and two other approaches to unawareness: that of Fagin and Halpern and that of Modica and Rustichini. We show that the differences between the HMS approach and the others are mainly due to the notion of validity used and the fact that the HMS is based on a 3-valued propositional logic.

1 Introduction

Reasoning about knowledge has played a significant role in work in philosophy, economics, and distributed computing. Most of that work has used standard Kripke structures to model knowledge, where an agent knows a fact φ if φ is true in all the worlds that the agent considers possible. While this approach has proved useful for many applications, it suffers from a serious shortcoming, known as the *logical omniscience* problem (first observed and named by Hintikka [1962]): agents know all tautologies and know all the logical consequences of their knowledge. This seems inappropriate for resource-bounded agents and agents who are unaware of various concepts (and thus do not know logical tautologies involving those concepts). To take just one simple example, a novice investor may not be aware of the notion of the price-earnings ratio, although that may be relevant to the decision of buying a stock.

There has been a great deal of work on the logical omniscience problem (see [Fagin, Halpern, Moses, and Vardi 1995] for an overview). Of most relevance to this paper are approaches that have focused on (lack of) awareness. Fagin and Halpern [1988] (FH from now on) were the first to deal with lack of model omniscience explicitly in terms of awareness. They did so by introducing an explicit awareness operator. Since then, there has been a stream of papers on the topic in the economics literature (see, for example, [Modica and Rustichini 1994; Modica and Rustichini 1999; Dekel, Lipman, and Rustichini 1998]). In these papers, awareness is defined in terms of knowledge: an agent is aware of p if he either knows p or knows that he does not know p. All of them focused on the single-agent case. Recently, Heifetz, Meier, and Schipper [2003] (HMS from now on) have provided a multi-agent model for unawareness. In this paper, we consider how the HMS model compares to other work.

A key feature of the HMS approach (also present in the work of Modica and Rustichini [1999]—MR from now on) is that with each world or state is associated a (propositional) language. Intuitively, this is the language of concepts defined at that world. Agents may not be aware of all these concepts. The way that is modeled is that in all the states an agent considers possible at a state s, fewer concepts may be defined than are defined at state s. Because a proposition p may be undefined at a given state s, the underlying logic in HMS is best viewed as a 3-valued logic: a proposition p may be true, false, or undefined at a given state.

We consider two sound and complete axiomatizations for the HMS model, that differ with respect to the language used and the notion of validity. One axiomatization captures *weak validity*: a formula is weakly valid if it is never false (although it may be undefined). In the single-agent case, this axiomatization is identical to that given by MR. However, in the MR model, validity is taken with respect to "objective" state, where all formulas are defined. As shown by Halpern [2001], this axiomatization is also sound and complete in the single-agent case with respect to a special case of FH's awareness structures; we extend Halpern's result to the multi-agent case. The other axiomatization of the HMS model captures *(strong) validity*: a formula is (strongly) valid if it is always true. If we add an axiom saying that there is no third value to this axiom system, then we just get the standard axiom system for S5. This shows that, when it comes to strong validity, the only difference between the HMS models and standard epistemic models is the third truth value.

The rest of this paper is organized as follows. In Section 2, we review the basic S5 model, the FH model, the MR model, and the HMS model. In Section 3, we compare the HMS approach and the FH approach, both semantically and axiomatically, much as Halpern [2001] compares the MR and FH approaches. We show that weak validity in HMS structures corresponds in a precise sense to validity in awareness structures. In Section 4, we extend the HMS language by adding a nonstandard implication operator. Doing so allows us to provide an axiomatization for strong validity. We conclude in Section 5. Further discussion of the original HMS framework and an axiomatization of strong validity in the purely propositional case can be found in the appendix.

2 Background

We briefly review the standard epistemic logic and the approaches of FH, MR, and HMS here.

2.1 Standard epistemic logic

The syntax of standard epistemic logic is straightforward. Given a set $\{1, \ldots, n\}$ of agents, formulas are formed by starting with a set $\Phi = \{p, q, \ldots\}$ of primitive propositions as well as a special formula $\top$ (which is always true), and then closing off under conjunction ($\wedge$), negation ($\neg$) and the modal operators K_i, $i = 1, \ldots, n$. Call the resulting language $\mathcal{L}_n^K(\Phi)$.[1] As usual, we define $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ as abbreviations of $\neg(\neg\varphi \wedge \neg\psi)$ and $\neg\varphi \vee \psi$, respectively.

The standard approach to giving semantics to $\mathcal{L}_n^K(\Phi)$ uses Kripke structures. A *Kripke structure for n agents (over Φ)* is a tuple $M = (\Sigma, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n)$, where Σ is a set of states, $\pi : \Sigma \times \Phi \rightarrow \{0, 1\}$ is an interpretation, which associates with each primitive propositions its truth value at each state in Σ, $\mathcal{K}_i : \Sigma \rightarrow 2^{\Sigma}$ is a *possibility correspondence* for agent i. Intuitively, if $t \in \mathcal{K}_i(s)$, then agent i considers state t possible at state s. We say $\mathcal{K}_i$ is *reflexive* if $\forall s \in \Sigma$, $s \in \mathcal{K}_i(s)$; it is *transitive* if $\forall s, t \in \Sigma$, if $t \in \mathcal{K}_i(s)$ then $\mathcal{K}_i(t) \subseteq \mathcal{K}_i(s)$; it is *Euclidean* if $\forall s, t \in \Sigma$, if $t \in \mathcal{K}_i(s)$ then $\mathcal{K}_i(t) \supseteq \mathcal{K}_i(s)$.[2] A Kripke structure is reflexive (resp., reflexive and transitive; partitional) if the possibility correspondences $\mathcal{K}_i$ are reflexive (resp., reflexive and transitive; reflexive, Euclidean, and transitive). Let $\mathcal{M}_n(\Phi)$ denote the class of all Kripke structures for n agents over Φ, with no restrictions on the $\mathcal{K}_i$ relations. We use the superscripts r, e, and t to indicate that the $\mathcal{K}_i$ relations are restricted to being reflexive, Euclidean, and transitive, respectively. Thus, for example, $\mathcal{M}_n^{rt}(\Phi)$ is the class of all reflexive and transitive Kripke structures for n agents.

[1] In MR, only the single-agent case is considered. We consider the multi-agent here to allow the generalization to HMS. In many cases, $\top$ is defined in terms of other formulas, e.g., as $\neg(p \wedge \neg p)$. We take it to be primitive here for convenience.

[2] It is more standard in the philosophy literature to take $\mathcal{K}_i$ to be a binary relation. The two approaches are equivalent, since if $\mathcal{K}_i'$ is a binary relation, we can define a possibility correspondence $\mathcal{K}_i$ by taking $t \in \mathcal{K}_i(s)$ iff $(s, t) \in \mathcal{K}_i'$. We can similarly define a binary relation given a possibility correspondence. Given this equivalence, it is easy to see that the notions of a possibility relation being reflexive, transitive, or Euclidean are equivalent to the corresponding notion for binary relations.

We write $(M,s) \models \varphi$ if φ is true at state s in the Kripke structure M. The truth relation is defined inductively as follows:

$$
\begin{array}{l}
(M,s) \models p, \text{ for } p \in \Phi, \text{ if } \pi(s,p) = 1 \\
(M,s) \models \neg\varphi \text{ if } (M,s) \not\models \varphi \\
(M,s) \models \varphi \wedge \psi \text{ if } (M,s) \models \varphi \text{ and } (M,s) \models \psi \\
(M,s) \models K_i\varphi \text{ if } (M,s') \models \varphi \text{ for all } s' \in \mathcal{K}_i(s).
\end{array}
$$

A formula φ is said to be *valid* in Kripke structure M if $(M,s) \models \varphi$ for all $s \in \Sigma$. A formula is valid in a class $\mathcal{N}$ of Kripke structures if it is valid for all Kripke structures in $\mathcal{N}$.

An *axiom system* AX consists of a collection of *axioms* and *inference rules*. An axiom is a formula, and an inference rule has the form "from $\varphi_1, \ldots, \varphi_k$ infer ψ," where $\varphi_1, \ldots, \varphi_k, \psi$ are formulas. A formula φ is *provable* in AX, denoted AX $\vdash \varphi$, if there is a sequence of formulas such that the last one is φ, and each one is either an axiom or follows from previous formulas in the sequence by an application of an inference rule. An axiom system AX is said to be *sound* for a language $\mathcal{L}$ with respect to a class $\mathcal{N}$ of structures if every formula provable in AX is valid with respect to $\mathcal{N}$. The system AX is *complete* for $\mathcal{L}$ with respect to $\mathcal{N}$ if every formula in $\mathcal{L}$ that is valid with respect to $\mathcal{N}$ is provable in AX.

Consider the following set of well-known axioms and inference rules:

Prop. All substitution instances of valid formulas of propositional logic.

K. $(K_i\varphi \wedge K_i(\varphi \Rightarrow \psi)) \Rightarrow K_i\psi$

T. $K_i\varphi \Rightarrow \varphi$

4. $K_i\varphi \Rightarrow K_iK_i\varphi$

5. $\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$

MP. From φ and $\varphi \Rightarrow \psi$ infer ψ (modus ponens)

Gen. From φ infer $K_i\varphi$

It is well known that the axioms T, 4, and 5 correspond to the requirements that the $\mathcal{K}_i$ relations are reflexive, transitive, and Euclidean, respectively. Let $\mathbf{K}_n$ be the axiom system consisting of the axioms Prop, K and rules MP, and Gen, and let $\mathbf{S5}_n$ be the system consisting of all the axioms and inference rules above. The following result is well known (see, for example, [Chellas 1980; Fagin, Halpern, Moses, and Vardi 1995] for proofs).

Theorem 2.1: *Let $\mathcal{C}$ be a (possibly empty) subset of $\{\mathrm{T}, 4, 5\}$ and let C be the corresponding subset of $\{r, t, e\}$. Then $\mathbf{K}_n \cup \mathcal{C}$ is a sound and complete axiomatization of the language $\mathcal{L}_n^K(\Phi)$ with respect to $\mathcal{M}_n^C(\Phi)$.*

In particular, this shows that $\mathbf{S5}_n$ characterizes partitional models, where the possibility correspondences are reflexive, transitive, and Euclidean.

2.2 The FH model

The Logic of General Awareness model of Fagin and Halpern [1988] introduces a syntactic notion of awareness. This is reflected in the language by adding a new modal operator A_i for each agent i. The intended interpretation of $A_i\varphi$ is "i is aware of φ". The power of this approach comes from the flexibility of the notion of awareness. For example, "agent i is aware of φ" may be interpreted as "agent i is familiar with all primitive propositions in φ" or as "agent i can compute the truth value of φ in time t".

Having awareness in the language allows us to distinguish two notions of knowledge: implicit knowledge and explicit knowledge. Implicit knowledge, denoted with K_i, is defined as truth in all worlds the agent considers possible, as usual. Explicit knowledge, denoted with X_i, is defined as the conjunction of implicit knowledge and awareness. Let $\mathcal{L}_n^{K,X,A}(\Phi)$ be the language extending $\mathcal{L}_n^K(\Phi)$ by closing off under the operators A_i and X_i, for $i = 1, \dots, n$. Let $\mathcal{L}_n^{X,A}(\Phi)$ (resp. $\mathcal{L}_n^X(\Phi)$) be the sublanguage of $\mathcal{L}_n^{K,X,A}(\Phi)$ where the formulas do not mention $K_1, \dots, K_n$ (resp., $K_1, \dots, K_n$ and $A_1, \dots A_n$).

An *awareness structure for n agents over Φ* is a tuple $M = (\Sigma, \pi, \mathcal{K}_1, ..., \mathcal{K}_n, \mathcal{A}_1, ..., \mathcal{A}_n)$, where $(\Sigma, \pi, \mathcal{K}_1, ..., \mathcal{K}_n)$ is a Kripke structure and $\mathcal{A}_i$ is a function associating a set of formulas for each state, for $i = 1, ..., n$. Intuitively, $\mathcal{A}_i(s)$ is the set of formulas that agent i is aware of at state s. The set of formulas the agent is aware of can be arbitrary. Depending on the interpretation of awareness one has in mind, certain restrictions on $\mathcal{A}_i$ may apply. There are two restrictions that are of particular interest here:

- Awareness is *generated by primitive propositions* if, for all agents i, $\varphi \in \mathcal{A}_i(s)$ iff all the primitive propositions that appear in φ are in $\mathcal{A}_i(s) \cap \Phi$. That is, an agent is aware of φ iff she is aware of all the primitive propositions that appear in φ.
- *Agents know what they are aware of* if, for all agents i, $t \in \mathcal{K}_i(s)$ implies that $\mathcal{A}_i(s) = \mathcal{A}_i(t)$.

Following Halpern [2001], we say that awareness structure is *propositionally determined* if awareness is generated by primitive propositions and agents know what they are aware of.

The semantics for awareness structures extends the semantics defined for standard Kripke structures by adding two clauses defining A_i and X_i:

$$(M, s) \models A_i\varphi \text{ if } \varphi \in \mathcal{A}_i(s)$$
$$(M, s) \models X_i\varphi \text{ if } (M, s) \models A_i\varphi \text{ and } (M, s) \models K_i\varphi.$$

FH provide a complete axiomatization for the logic of awareness; we omit the details here.

2.3 The MR model

We follow Halpern's [2001] presentation of MR here; it is easily seen to be equivalent to that in [Modica and Rustichini 1999].

Since MR consider only the single-case, they use the language $\mathcal{L}_1^K(\Phi)$. A *generalized standard model* (GSM) over Φ has the form $M = (S, \Sigma, \pi, \mathcal{K}, \rho)$, where

- S and Σ are disjoint sets of states; moreover, $\Sigma = \cup_{\Psi \subseteq \Phi} S_\Psi$, where the sets S_Ψ are disjoint. Intuitively, the states in S describe the objective situation, while the states in Σ describe the agent's subjective view of the objective situation, limited to the vocabulary that the agent is aware of.
- $\pi : S \times \Phi \Rightarrow \{0, 1\}$ is an interpretation.
- $\mathcal{K} : S \to 2^\Sigma$ is a *generalized possibility correspondence*.

- ρ is a projection from S to Σ such that (1) if $\rho(s) = \rho(t) \in S_\Psi$ then (a) s and t agree on the truth values of all primitive propositions in Ψ, that is, $\pi(s,p) = \pi(t,p)$ for all $p \in \Psi$ and (b) $\mathcal{K}(s) = \mathcal{K}(t)$ and (2) if $\rho(s) \in S_\Psi$, then $\mathcal{K}(s) \subseteq S_\Psi$. Intuitively, $\rho(s)$ is the agent's subjective state in objective state s.

We can extend $\mathcal{K}$ to a map (also denoted $\mathcal{K}$ for convenience) defined on $S \cup \Sigma$ in the following way: if $s' \in \Sigma$ and $\rho(s) = s'$, define $\mathcal{K}(s') = \mathcal{K}(s)$. Condition 1(b) on ρ guarantees that this extension is well defined. A GSM is reflexive (resp., reflexive and transitive; partitional) if $\mathcal{K}$ restricted to Σ is reflexive (resp., reflexive and transitive; reflexive, Euclidean and transitive). Similarly, we can extend π to a function (also denoted π) defined on $S \cup \Sigma$: if $s' \in S_\Psi$, $p \in \Psi$ and $\rho(s) = s'$, define $\pi(s',p) = \pi(s,p)$; and if $s' \in S_\Psi$ and $p \notin \Psi$, define $\pi(s',p) = 1/2$.

With these extensions of $\mathcal{K}$ and π, the semantics for formulas in GSMs is identical to that in standard Kripke structures except for the negation, which is defined as follows:

$$\text{if } s \in S, \text{ then } (M,s) \models \neg\varphi \text{ iff } (M,s) \not\models \varphi$$
$$\text{if } s \in S_\Psi, \text{ then } (M,s) \models \neg\varphi \text{ iff } (M,s) \not\models \varphi \text{ and } \varphi \in \mathcal{L}_1^K(\Psi).$$

Note that for states in the "objective" state space S, the logic is 2-valued; and every formula is either true or false. On the other hand, for states in the "subjective" state space Σ the logic is 3-valued. A formula may be neither true nor false. It is easy to check that if $s \in S_\Psi$, then every formula in $\mathcal{L}_1^K(\Psi)$ is either true or false at s, while formulas not in $\mathcal{L}_1^K(\Psi)$ are neither true nor false. Intuitively, an agent can assign truth values only to formulas involving concepts he is aware of; at states in S_Ψ, the agent is aware only of concepts expressed in the language $\mathcal{L}_1^K(\Psi)$.

The intuition behind MR's notion of awareness is that an agent is unaware of φ if he does not know φ, does not know he does not know it, and so on. Thus, an agent is aware of φ if he either knows φ or knows he does not know φ, or knows that he does not know that he does not know φ, or .... MR show that under appropriate assumptions, this infinite disjunction is equivalent to the first two disjuncts, so they define $A\varphi$ to be an abbreviation of $K\varphi \vee K\neg K\varphi$.

Rather than considering validity, MR consider what we call here *objective validity*: truth in all objective states (that is, the states in S). Note that all classical (2-valued) propositional tautologies are objectively valid in the MR setting. MR provide a system $\mathcal{U}$ that is a sound and complete axiomatization for objective validity with respect to partitional GSM structures. The system $\mathcal{U}$ consists of the axioms Prop, T, and 4, the inference rule MP, and the following additional axioms and inference rules:

M $K(\varphi \wedge \psi) \Rightarrow K\varphi \wedge K\psi$

C $K\varphi \wedge K\psi \Rightarrow K(\varphi \wedge \psi)$

A. $A\varphi \Leftrightarrow A\neg\varphi$

AM. $A(\varphi \wedge \psi) \Rightarrow A\varphi \wedge A\psi$

N. $K\top$

RE$_{sa}$. From $\varphi \Leftrightarrow \psi$ infer $K\varphi \Leftrightarrow K\psi$, where φ and ψ contain exactly the same primitive propositions.

Theorem 2.2: [Modica and Rustichini 1999] *$\mathcal{U}$ is a complete and sound axiomatization of objective validity for the language $\mathcal{L}_1^K(\Phi)$ with respect to partitional GSMs over Φ.*

2.4 The HMS model

HMS define their approach semantically, without giving a logic. We discuss their semantic approach in the appendix. To facilitate comparison of HMS to the other approaches we have considered, we define an appropriate logic. (In recent work done independently of ours [Heifetz, Meier, and Schipper 2005], HMS also consider a logic based on their approach, whose syntax and semantics is essentially identical to that described here.)

Given a set Φ of primitive propositions, consider again the language $\mathcal{L}_n^K(\Phi)$. An *HMS structure for n agents* (over Φ) is a tuple $M = (\Sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n, \pi, \{\rho_{\Psi',\Psi} : \Psi \subseteq \Psi' \subseteq \Phi\})$, where (as in MR), $\Sigma = \cup_{\Psi \subseteq \Phi} S_\Psi$ is a set of states, $\pi : \Sigma \times \Phi \rightarrow \{0, 1, 1/2\}$ is an interpretation such that for $s \in S_\Psi$, $\pi(s, p) \neq 1/2$ iff $p \in \Psi$ (intuitively, all primitive propositions in Ψ are defined at states of S_Ψ), and $\rho_{\Psi',\Psi}$ maps $S_{\Psi'}$ onto S_Ψ. Intuitively, $\rho_{\Psi',\Psi}(s)$ is a description of the state $s \in S_{\Psi'}$ in the less expressive vocabulary of S_Ψ. Moreover, if $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$, then $\rho_{\Psi_3,\Psi_2} \circ \rho_{\Psi_2,\Psi_1} = \rho_{\Psi_3,\Psi_1}$. Note that although both MR and HMS have projection functions, they have slightly different intuitions behind them. For MR, $\rho(s)$ is the subjective state (i.e., the way the world looks to the agent) when the actual objective state is s. For HMS, there is no objective state; $\rho_{\Psi',\Psi}(s)$ is the description of s in the less expressive vocabulary of S_Ψ. Finally, the $\models$ relation in HMS structures is defined for formulas in $\mathcal{L}_n^K(\Phi)$ in exactly the same way as it is in subjective states of MR structures. Moreover, like MR, $A_i\varphi$ is defined as an abbreviation of $K_i\varphi \vee K_i\neg K_i\varphi$.

Note that the definition of $\models$ does not use the functions $\rho_{\Psi,\Psi'}$. These functions are used only to impose some coherence conditions on HMS structures. To describe these conditions, we need a definition. Given $B \subseteq S_\Psi$, let $B^\uparrow = \cup_{\Psi' \supseteq \Psi} \rho_{\Psi',\Psi}^{-1}(B)$. Thus, we can think of $B^\uparrow$ as the states in which B can be expressed.

1. Confinedness: If $s \in S_\Psi$ then $\mathcal{K}_i(s) \subseteq S_{\Psi'}$ for some $\Psi' \subseteq \Psi$.
2. Generalized reflexivity: $s \in \mathcal{K}_i(s)^\uparrow$ for all $s \in \Sigma$.
3. Stationarity: $s' \in \mathcal{K}_i(s)$ implies
   (a) $\mathcal{K}_i(s') \subseteq \mathcal{K}_i(s)$;
   (b) $\mathcal{K}_i(s') \supseteq \mathcal{K}_i(s)$.
4. Projections preserve knowledge: If $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$, $s \in S_{\Psi_3}$, and $\mathcal{K}_i(s) \subseteq S_{\Psi_2}$, then $\rho_{\Psi_2,\Psi_1}(\mathcal{K}_i(s)) = \mathcal{K}_i(\rho_{\Psi_3,\Psi_1}(s))$.
5. Projections preserve ignorance: If $s \in S_{\Psi'}$ and $\Psi \subseteq \Psi'$ then $(\mathcal{K}_i(s))^\uparrow \subseteq (\mathcal{K}_i(\rho_{\Psi',\Psi}(s)))^\uparrow$.[3]

We remark that HMS combined parts (a) and (b) of stationarity into one statement (saying $\mathcal{K}_i(s) = \mathcal{K}_i(s')$). We split the condition in this way to make it easier to capture axiomatically. Roughly speaking, generalized reflexivity, part (a) of stationarity, and part (b) of stationarity are analogues of the assumptions in standard epistemic structures that the possibility correspondences are reflexive, transitive, and Euclidean, respectively. The remaining assumptions can be viewed as coherence conditions. See [Heifetz, Meier, and Schipper 2003] for further discussion of these conditions.

If C is a subset of $\{r, t, e\}$, let $\mathcal{H}_n^C(\Phi)$ denote the class of HMS structures over Φ satisfying confinedness, projections preserve knowledge, projections preserve ignorance, and the subset of generalized reflexivity, part (a) of stationarity, and part (b) of stationarity corresponding to C. Thus, for example, $\mathcal{H}_n^{rt}(\Phi)$ is the class of HMS structures for n agents over Φ that satisfy confinedness, projections preserve knowledge, projections preserve ignorance, generalized reflexivity, and part (a) of stationarity. HMS consider only

[3] HMS explicitly assume that $\mathcal{K}_i(s) \neq \emptyset$ for all $s \in \Sigma$, but since this follows from generalized reflexivity we do not assume it explicitly. HMS also mention one other property, which they call *projections preserve awareness*, but, as HMS observe, it follows from the assumption that projections preserve knowledge, so we do not consider it here.

"partitional" HMS structures, that is, structures in $\mathcal{H}_n^{rte}(\Phi)$. However, we can get more insight into HMS structures by allowing the greater generality of considering non-partitional structures.

3 A Comparison of the Approaches

As a first step to comparing the MR, HMS, and FH approaches, we recall a result proved by Halpern.

Lemma 3.1: [Halpern 2001, Lemma 2.1] *If M is a partitional awareness structures where awareness is generated by primitive propositions, then*

$$M \models A_i\varphi \Leftrightarrow (X_i\varphi \vee (\neg X_i\varphi \wedge X_i\neg X_i\varphi)).$$

Halpern proves this lemma only for the single-agent case, but the proof goes through without change for the multi-agent case. Note that this equivalence does not hold in general in non-partitional structures.

Thus, if we restrict to partitional awareness structures where awareness is generated by primitive propositions, we can define awareness just as MR and HMS do.

Halpern [2001, Theorem 4.1] proves an even stronger connection between the semantics of FH and MR, essentially showing that partitional GSMs are in a sense equivalent to propositionally determined awareness structures. We prove a generalization of this result here.

If C is a subset of $\{r,t,e\}$, let $\mathcal{N}_n^{C,pd}(\Phi)$ denote the set of propositionally determined awareness structures over Φ whose $\mathcal{K}_i$ relations satisfy the conditions in C. Given a formula $\varphi \in \mathcal{L}_n^K(\Phi)$, let $\varphi_X \in \mathcal{L}_n^X(\Phi)$ be the formula that results by replacing all occurrences of K_i in φ by X_i. Finally, let Φ_φ be the set of primitive propositions appearing in φ.

Theorem 3.2: *Let C be a subset of $\{r,t,e\}$ that contains at least one of t or e.*

- *(a) If $M = (\Sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n, \pi, \{\rho_{\Psi',\Psi} : \Psi \subseteq \Psi' \subseteq \Phi\} \in \mathcal{H}_n^C(\Phi)$, then there exists an awareness structure $M' = (\Sigma, \mathcal{K}_1', \ldots, \mathcal{K}_n', \pi', \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathcal{N}_n^{C,pd}(\Phi)$ such that if $s \in S_\Psi$ and $\Phi_\varphi \subseteq \Psi$, then $(M,s) \models \varphi$ iff $(M',s) \models \varphi_X$.*
- *(b) If $M = (\Sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n, \pi, \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathcal{N}_n^{C,pd}(\Phi)$, then there exists an HMS structure $M' = (\Sigma', \mathcal{K}_1', \ldots, \mathcal{K}_n', \pi', \{\rho_{\Psi',\Psi} : \Psi \subseteq \Psi' \subseteq \Phi\}) \in \mathcal{H}_n^C(\Phi)$ such that $\Sigma' = \Sigma \times 2^\Phi$, $S_\Psi = \Sigma \times \{\Psi\}$ for all $\Psi \subseteq \Phi$, and if $\Phi_\varphi \subseteq \Psi$, then $(M,s) \models \varphi_X$ iff $(M',(s,\Psi)) \models \varphi$.*

It follows immediately from Halpern's analogue of Theorem 3.2 that φ is objectively valid in GSMs iff φ_X is valid in propositionally determined partitional awareness structures. Thus, objective validity in GSMs and validity in propositionally determined partitional awareness structures are characterized by the same set of axioms.

We would like to get a similar result here. However, if we define validity in the usual way—that is, φ is valid iff $(M,s) \models \varphi$ for all states s and all HMS structures M—then it is easy to see that there are no (non-trivial) valid HMS formulas. Since the HMS logic is three-valued, besides what we will call *strong validity* (truth in all states), we can consider another standard notion of validity. A formula is *weakly valid* iff it is not false at any state in any HMS structure (that is, it is either true or undefined at every state in every HMS structure). Put another way, φ is weakly valid if, at all states where φ is defined, φ is true.

Corollary 3.3: *Let C be a subset of $\{r,t,e\}$ that contains at least one of t or e. Then φ is weakly valid in $\mathcal{H}_n^C(\Phi)$ iff φ_X is valid in $\mathcal{N}_n^{C,pd}(\Phi)$.*

Halpern [2001] provides a sound and complete axiomatizations for the language $\mathcal{L}_1^{X,A}(\Phi)$ with respect to $\mathcal{N}^{C,pd}(\Phi)$, where C is either $\emptyset$, $\{r\}$, $\{r,t\}$ and $\{r,e,t\}$. It is straightforward to extend his techniques to other subsets of $\{r,e,t\}$ and to arbitrary numbers of agents. However, these axioms involve combinations of X_i and A_i; for example, all the systems have an axiom of the form $X\varphi \wedge X(\varphi \Rightarrow \psi) \wedge A\psi \Rightarrow X\psi$. There seems to be no obvious axiomatization for $\mathcal{L}_n^X(\Phi)$ that just involves axioms in the language $\mathcal{L}_n^X(\Phi)$ except for the special case of partitional awareness structures, where A_i is definable in terms of X_i (see Lemma 3.1), although this may simply be due to the fact that there are no interesting axioms for this language.

Let $\mathrm{S5}_n^X$ be the n-agent version of the axiom system $\mathrm{S5}_X$ that Halpern proves is sound and complete for $\mathcal{L}_n^X(\Phi)$ with respect to $\mathcal{N}_n^{ret,pd}(\Phi)$ (so that, for example, the axiom $X\varphi \wedge X(\varphi \Rightarrow \psi) \wedge A\psi \Rightarrow X\psi$ becomes $X_i\varphi \wedge X_i(\varphi \Rightarrow \psi) \wedge A_i\psi \Rightarrow X_i\psi$, where now we view $A_i\varphi$ as an abbreviation for $X_i\varphi \vee X_i\neg X_i\varphi$). Let $\mathrm{S5}_n^K$ be the result of replacing all occurrences of X_i in formulas in $\mathrm{S5}_n^X$ by K_i. Similarly, let $\mathcal{U}_n$ be the n-agent version of the axiom system $\mathcal{U}$ together with the axiom $A_iK_j\varphi \Leftrightarrow A_i\varphi$, [4] and let $\mathcal{U}_n^X$ be the result of replacing all instances of K_i in the axioms of $\mathcal{U}_n$ by X_i. HMS have shown that there is a sense in which a variant of $\mathcal{U}_n$ (which is easily seen to be equivalent to $\mathcal{U}_n$) is a sound and complete axiomatization for HMS structures [Heifetz, Meier, and Schipper 2005]. Although this is not the way they present it, their results actually show that $\mathcal{U}_n$ is a sound and complete axiomatization of weak validity with respect to $\mathcal{H}_n^{ret}(\Phi)$.

Thus, the following is immediate from Corollary 3.3.

Corollary 3.4: *$\mathcal{U}_n$ and $\mathrm{S5}_n^K$ are both sound and complete axiomatization of weak validity for the language $\mathcal{L}_n^K(\Phi)$ with respect to $\mathcal{H}_n^{ret}(\Phi)$; $\mathcal{U}_n^X$ and $\mathrm{S5}_n^X$ are both sound and complete axiomatizations of validity for the language $\mathcal{L}_n^X(\Phi)$ with respect to $\mathcal{N}_n^{ret,pd}(\Phi)$.*

We can provide a direct proof that $\mathcal{U}_n$ and $\mathrm{S5}_n^K$ (resp., $\mathcal{U}_n^X$ and $\mathrm{S5}_n^X$) are equivalent, without appealing to Corollary 3.3. It is easy to check that all the axioms of $\mathcal{U}_n^X$ are valid in $\mathcal{N}_n^{ret,pd}(\Phi)$ and all the inference rules of $\mathcal{U}_n^X$ preserve validity. From the completeness of $\mathrm{S5}_n^X$ proved by Halpern, it follows that anything provable in $\mathcal{U}_n^X$ is provable in $\mathrm{S5}_n^X$, and hence that anything provable in $\mathcal{U}_n$ is provable in $\mathrm{S5}_n^K$. Similarly, it is easy to check that all the axioms of $\mathrm{S5}_n^K$ are weakly valid in $\mathcal{H}_n^{ret}(\Phi)$, and the inference rules preserve validity. Thus, from the results of HMS, it follows that everything provable in $\mathrm{S5}_n^K$ is provable in $\mathcal{U}_n$ (and hence that everything provable in $\mathrm{S5}_n^X$ is provable in $\mathcal{U}_n^X$).

These results show a tight connection between the various approaches. $\mathcal{U}$ is a sound and complete axiomatization for objective validity in partitional GSMs; $\mathcal{U}_n$ is a sound and complete axiomatization for weak validity in partitional HMS structures; and $\mathcal{U}_n^X$ is a sound and complete axiomatization for (the standard notion of) validity in partitional awareness structures where awareness is generated by primitive propositions and agents know which formulas they are aware of.

4 Strong Validity

We say a formula is *(strongly) valid* in HMS structures if it is true at every state in every HMS structure. We can get further insight into HMS structures by considering strong validity. However, since no nontrivial formulas in $\mathcal{L}_n^K(\Phi)$ are valid in HMS structures, we must first extend the language. We do so by adding a nonstandard implication operator $\hookrightarrow$ to the language.[5] Given an HMS structure M, define $[\![\varphi]\!]_M = \{s : (M,s) \models \varphi\}$; that is, $[\![\varphi]\!]_M$ is the set of states in M where φ is true. Roughly speaking, we want to define $\hookrightarrow$ in such a way that if $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$, then $\varphi \hookrightarrow \psi$ is valid in M. The one time when we do not necessarily want this is if $\varphi_M = \emptyset$. For example, we definitely do not want $r \vee (p \wedge \neg p) \hookrightarrow r \vee (q \wedge \neg q)$ to be valid

[4] The single-agent version of this axiom, $AK\varphi \Leftrightarrow A\varphi$, is provable in $\mathcal{U}$, so does not have to be given separately.

[5] We remark that a nonstandard implication operator was also added to the logic used by Fagin, Halpern, and Vardi [1995] for exactly the same reason, although the semantics of the operator here is different from there, since the underlying logic is different.

(since $r \vee (p \wedge \neg p)$ will be true at a state where r is true, p is defined, and q is undefined, while $r \wedge (q \wedge \neg q)$ is undefined at such a state). Thus, it seems unreasonable to have $p \wedge \neg p \hookrightarrow q \wedge \neg q$ be valid, even though $[\![p \wedge \neg p]\!]_M = \emptyset$. If $[\![\varphi]\!]_M = \emptyset$, we take $\varphi \hookrightarrow \psi$ to be valid only if ψ is at least as defined as φ. Since the set of states where ψ is defined in M is $[\![\psi]\!]_M \cup [\![\neg\psi]\!]_M$, this condition becomes $[\![\neg\varphi]\!]_M \subseteq [\![\psi]\!]_M \cup [\![\neg\psi]\!]_M$.

Let $\mathcal{L}_n^{K,\hookrightarrow}(\Phi)$ be the language that results by closing off under $\hookrightarrow$ in addition to $\neg$, $\wedge$, and $K_1, \ldots, K_n$; let $\mathcal{L}^{\hookrightarrow}(\Phi)$ be the propositional fragment of the language. We cannot use the MR definition of negation for $\mathcal{L}_n^{K,\hookrightarrow}(\Phi)$, since $\varphi \hookrightarrow \psi$ may be defined even in states where φ and ψ are not defined. (For example, $p \hookrightarrow p$ is true in all states, even in states where p is not defined.) Thus, we must separately define the truth and falsity of all formulas at all states, which we do as follows. In the definitions, we use $(M,s) \models\uparrow\varphi$ as an abbreviation of $(M,s) \not\models \varphi$ and $(M,s) \not\models \neg\varphi$; and $(M,s) \models\downarrow\varphi$ as an abbreviation of $(M,s) \models \varphi$ or $(M,s) \models \neg\varphi$ (so $(M,s) \models\uparrow\varphi$ iff φ is neither true nor false at s, i.e., it is undefined at s).

$(M,s) \models \top$
$(M,s) \not\models \neg\top$
$(M,s) \models p$ if $\pi(s,p) = 1$
$(M,s) \models \neg p$ if $\pi(s,p) = 0$
$(M,s) \models \neg\neg\varphi$ if $(M,s) \models \varphi$
$(M,s) \models \varphi \wedge \psi$ if $(M,s) \models \varphi$ and $(M,s) \models \psi$
$(M,s) \models \neg(\varphi \wedge \psi)$ if either $(M,s) \models \neg\varphi \wedge \psi$ or $(M,s) \models \varphi \wedge \neg\psi$ or $(M,s) \models \neg\varphi \wedge \neg\psi$
$(M,s) \models (\varphi \hookrightarrow \psi)$ if either $(M,s) \models \varphi \wedge \psi$ or $(M,s) \models\uparrow\varphi$ or $(M,s) \models \neg\varphi \wedge \downarrow\psi$
$(M,s) \models \neg(\varphi \hookrightarrow \psi)$ if $(M,s) \models \varphi$ and $(M,s) \models \neg\psi$
$(M,s) \models K_i\varphi$ if $(M,s) \models\downarrow\varphi$ and $(M,t) \models \varphi$ for all $t \in \mathcal{K}_i(s)$
$(M,s) \models \neg K_i\varphi$ if $(M,s) \not\models K_i\varphi$ and $(M,s) \models\downarrow\varphi$.

It is easy to check that this semantics agrees with the MR semantics for formulas in $\mathcal{L}_n^K(\Phi)$. Moreover, the following lemma follows by an easy induction on the structure of formulas.

Lemma 4.1: *If $\Psi \subseteq \Psi'$, every formula in $\mathcal{L}_n^{K,\hookrightarrow}(\Psi)$ is defined at every state in $S_{\Psi'}$.*

It is useful to define the following abbreviations:

- $\varphi \rightleftharpoons \psi$ is an abbreviation of $(\varphi \hookrightarrow \psi) \wedge (\psi \hookrightarrow \varphi)$;
- $\varphi = 1$ is an abbreviation of $\neg(\varphi \hookrightarrow \neg\top)$;
- $\varphi = 0$ is an abbreviation of $\neg(\neg\varphi \hookrightarrow \neg\top)$;
- $\varphi = \frac{1}{2}$ is an abbreviation of $(\varphi \hookrightarrow \neg\top) \wedge (\neg\varphi \hookrightarrow \neg\top)$.

Using the formulas $\varphi = 0$, $\varphi = \frac{1}{2}$, and $\varphi = 1$, we can reason directly about the truth value of formulas. This will be useful in our axiomatization.

In our axiomatization of $\mathcal{L}_n^{K,\hookrightarrow}(\Phi)$ with respect to HMS structures, just as in standard epistemic logic, we focus on axioms that characterize properties of the $\mathcal{K}_i$ relation that correspond to reflexivity, transitivity, and the Euclidean property.

Consider the following axioms:

Prop′. All substitution instances of formulas valid in $\mathcal{L}^{\hookrightarrow}(\Phi)$.

K$'$. $K_i\varphi \wedge K_i(\varphi \hookrightarrow \psi)) \hookrightarrow K_i\psi$

T$'$. $K_i\varphi \hookrightarrow \varphi \vee \bigvee_{p:p\in\Phi_\varphi} K_i(p = 1/2)$.

4$'$. $K_i\varphi \hookrightarrow K_iK_i\varphi$.

5$'$. $\neg K_i\neg K_i\varphi \hookrightarrow (K_i\varphi) \vee K_i(\varphi = 1/2)$

Conf1. $(\varphi = 1/2) \hookrightarrow K_i(\varphi = 1/2)$ if $\varphi \in \mathcal{L}_n^K(\Phi)$

Conf2. $\neg K_i(\varphi = 1/2) \hookrightarrow K_i((\varphi \vee \neg\varphi) = 1)$

B1. $(K_i\varphi) = 1/2 \rightleftharpoons \varphi = 1/2$

B2. $(K_i\varphi) = 1 \rightleftharpoons (K_i(\varphi = 1)) = 1$

MP$'$. From φ and $\varphi \hookrightarrow \psi$ infer ψ.

A few comments regarding the axioms: Prop$'$, K$'$, T$'$, 4$'$, 5$'$, and MP$'$ are weakenings of the corresponding axioms and inference rule for standard epistemic logic. All of them use $\hookrightarrow$ rather than $\Rightarrow$; in some cases further weakening is required. We provide an axiomatic characterization of Prop$'$ in the appendix. A key property of the axiomatization is that if we just add the axiom $\varphi \neq 1/2$ (saying that all formulas are defined), we get a complete axiomatization of classical logic. T (with $\Rightarrow$ replaced by $\hookrightarrow$) is sound in HMS systems satisfying generalized reflexivity for formulas φ in $\mathcal{L}_n^K(\Phi)$. But, for example, $K_i(p = 1/2) \hookrightarrow p = 1/2$ is not valid; p may be defined (i.e., be either true or false) at a state s and undefined at all states $s' \in \mathcal{K}_i(s)$. Note that axiom 5 is equivalent to its contrapositive $\neg K_i\neg K_i\varphi \Rightarrow K_i\varphi$. This is not sound in its full strength; for example, if p is defined at s but undefined at the states in $\mathcal{K}_i(s)$, then $(M, s) \models \neg K_i\neg K_ip \wedge \neg K_ip$. Axioms Conf1 and Conf2, as the names suggest, capture confinedness. We can actually break confinedness into two parts. If $s \in S_\Psi$, the first part says that each state $s' \in \mathcal{K}_i(s)$ is in some set $S_{\Psi'}$ such that $\Psi' \subseteq \Psi$. In particular, that means that a formula in $\mathcal{L}_n^K(\Phi)$ that is undefined at s must be undefined at each state in $\mathcal{K}_i(s)$. This is just what Conf1 says. Note that Conf1 does not hold for arbitrary formulas; for example, if p is defined and q is undefined at s, and both are undefined at all states in $\mathcal{K}_i(s)$, then $(M, s) \models (p \hookrightarrow q) = 1/2 \wedge \neg K_i((p \hookrightarrow q) = 1/2)$. The second part of confinedness says that all states in $\mathcal{K}_i(s)$ are in the same set $S_{\Psi'}$. This is captured by Conf2, since it says that if φ is defined at some state in $\mathcal{K}_i(s)$, then it is defined at all states in $\mathcal{K}_i(s)$. Finally, B1 and B2 are technical axioms that capture the semantics of $K_i\varphi$.

Let $\mathrm{AX}_\mathrm{n}^{\mathrm{K},\hookrightarrow}$ be the system consisting of Prop$'$, K$'$, B1, B2, Conf1, Conf2, MP$'$, and Gen.

Theorem 4.2: *Let $\mathcal{C}$ be a (possibly empty) subset of $\{\mathrm{T}', 4', 5'\}$ and let C be the corresponding subset of $\{r, t, e\}$. Then $\mathrm{AX}_\mathrm{n}^{\mathrm{K},\hookrightarrow} \cup \mathcal{C}$ is a sound and complete axiomatization of the language $\mathcal{L}_n^{K,\hookrightarrow}(\Phi)$ with respect to $\mathcal{H}_n^C(\Phi)$.*

Theorem 4.2 also allows us to relate HMS structures to standard epistemic structures. It is easy to check that if $\mathcal{C}$ is a (possibly empty) subset of $\{\mathrm{T}', 4', 5'\}$ and C is the corresponding subset of $\{r, e, t\}$, all the axioms of $\mathrm{AX}_\mathrm{n}^{\mathrm{K},\hookrightarrow} \cup \mathcal{C}$ are sound with respect to standard epistemic structures $\mathcal{M}_n^C(\Phi)$. Moreover, we get completeness by adding the axiom $\varphi \neq 1/2$, which says that all formulas are either true or false. Thus, in a precise sense, HMS differs from standard epistemic logic by allowing a third truth value.

5 Conclusion

We have compared the HMS approach and the FH approach to modeling unawareness. Our results show that, as long as we restrict to the language $\mathcal{L}_n^K(\Phi)$, the approaches are essentially equivalent; we can translate from one to the other. We are currently investigating extending the logic of awareness by allowing awareness of unawareness [Halpern and Rêgo], so that it would be possible to say, for example, that there exists a fact that agent 1 is unaware of but agent 1 knows that agent 2 is aware of it. This would be expressed by the formula $\exists p(\neg A_1 p \land X_1 A_2 p)$. Such reasoning seems critical to capture what is going on in a number of games. Moreover, it is not clear whether it can be expressed in the HMS framework.

Acknowledgments: We thank Aviad Heifetz, Martin Meier, and Burkhard Schipper for useful discussions on awareness. This work was supported in part by NSF under grants CTC-0208535 and ITR-0325453, by ONR under grants N00014-00-1-03-41 and N00014-01-10-511, and by the DoD Multidisciplinary University Research Initiative (MURI) program administered by the ONR under grant N00014-01-1-0795. The second author was also supported in part by a scholarship from the Brazilian Government through the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

References

Chellas, B. F. (1980). *Modal Logic*. Cambridge, U.K.: Cambridge University Press.

Dekel, E., B. Lipman, and A. Rustichini (1998). Standard state-space models preclude unawareness. *Econometrica 66*, 159–173.

Fagin, R. and J. Y. Halpern (1988). Belief, awareness, and limited reasoning. *Artificial Intelligence 34*, 39–76.

Fagin, R., J. Y. Halpern, Y. Moses, and M. Y. Vardi (1995). *Reasoning about Knowledge*. Cambridge, Mass.: MIT Press. A slightly revised paperback version was published in 2003.

Fagin, R., J. Y. Halpern, and M. Y. Vardi (1995). A nonstandard approach to the logical omniscience problem. *Artificial Intelligence 74*(2), 203–240.

Halpern, J. Y. (2001). Alternative semantics for unawareness. *Games and Economic Behavior 37*, 321–339.

Halpern, J. Y. and L. C. Rêgo. Reasoning about knowledge of unawareness. Unpublished manuscript.

Heifetz, A., M. Meier, and B. Schipper (2003). Multi-person unawareness. In *Theoretical Aspects of Rationality and Knowledge: Proc. Ninth Conference (TARK 2003)*, pp. 148–158. An extended version with the title "Interactive unawareness" is available at http://www.core.ucl.ac.be/services/psfiles/dp04/dp2004_59.pdf and will apear in the *Journal of Economic Theory*.

Heifetz, A., M. Meier, and B. Schipper (2005). A canonical model for interactive unawareness. Unpublished manuscript.

Hintikka, J. (1962). *Knowledge and Belief*. Ithaca, N.Y.: Cornell University Press.

Modica, S. and A. Rustichini (1994). Awareness and partitional information structures. *Theory and Decision 37*, 107–124.

Modica, S. and A. Rustichini (1999). Unawareness and partitional information structures. *Games and Economic Behavior 27*(2), 265–298.

A The Original HMS Approach

HMS describe their approach purely semantically, without giving a logic. We review their approach here (making some inessential changes for ease of exposition). An *HMS frame for n agents* is a tuple $(\Sigma, \mathcal{K}_1, \ldots, \mathcal{K}_n, (\Delta, \preceq), \{\rho_{\beta,\alpha} : \alpha, \beta \in \Delta, \alpha \preceq \beta\})$, where:

- Δ is an arbitrary lattice, partially ordered by $\preceq$;
- $\mathcal{K}_1, \ldots, \mathcal{K}_n$ are possibility correspondences, one for each agent;
- Σ is a disjoint union of the form $\cup_{\alpha \in \Delta} S_\alpha$;
- if $\alpha \preceq \beta$, then $\rho_{\beta,\alpha} : S_\beta \to S_\alpha$ is a surjection.

In the logic-based version of HMS given in Section 2.4, Δ consists of the subsets of Φ, and $\Psi \preceq \Psi'$ iff $\Psi \subseteq \Psi'$. Thus, the original HMS definition can be viewed as a more abstract version of that given in Section 2.4.

Given $B \subseteq S_\alpha$, let $B^\uparrow = \cup_{\{\beta:\, \alpha \preceq \beta\}} \rho_{\beta,\alpha}^{-1}(B)$. We can think of $B^\uparrow$ as the states in which B can be expressed. HMS focus on sets of the form $B^\uparrow$, which they take to be events.

HMS assume that their frames satisfy the five conditions mentioned in Section 2.4, restated in their more abstract setting. The statements of generalized reflexivity and stationarity remain the same. Confinedness, projections preserve knowledge, and projections preserve ignorance are stated as follows:

- confinedness: if $s \in S_\beta$ then $\mathcal{K}_i(s) \subseteq S_\alpha$ for some $\alpha \preceq \beta$;
- projections preserve knowledge: if $\alpha \preceq \beta \preceq \gamma$, $s \in S_\gamma$, and $\mathcal{K}_i(s) \subseteq S_\beta$, then $\rho_{\beta,\alpha}(\mathcal{K}_i(s)) = \mathcal{K}_i(\rho_{\gamma,\alpha}(s))$;
- projections preserve ignorance: if $s \in S_\beta$ and $\alpha \preceq \beta$ then $(\mathcal{K}_i(s))^\uparrow \subseteq (\mathcal{K}_i(\rho_{\beta,\alpha}(s)))^\uparrow$.

HMS start by considering the algebra consisting of all events of the form $B^\uparrow$. In this algebra, they define an operator $\neg$ by taking $\neg(B^\uparrow) = (S_\alpha - B)^\uparrow$ for $\emptyset \neq B \subseteq S_\alpha$. With this definition, $\neg\neg B^\uparrow = B^\uparrow$ if $B \notin \{\emptyset, S_\alpha\}$. However, it remains to define $\neg\emptyset^\uparrow$. We could just take it to be Σ, but then we have $\neg\neg S_\alpha^\uparrow = \Sigma$, rather than $\neg\neg S_\alpha^\uparrow = S_\alpha^\uparrow$. To avoid this problem, in their words, HMS "devise a distinct vacuous event $\emptyset^{S_\alpha}$" for each subspace S_α, extend the algebra with these events, and define $\neg S_\alpha^\uparrow = \emptyset^{S_\alpha}$ and $\neg\emptyset^{S_\alpha} = S_\alpha^\uparrow$. They do not make clear exactly what it means to "devise a vacuous event".We can recast their definitions in the following way, that allows us to bring in the events $\emptyset^{S_\alpha}$ more naturally.

In a 2-valued logic, given a formula φ and a structure M, the set $[\![\varphi]\!]_M$ of states where φ is true and the set $[\![\neg\varphi]\!]_M$ of states where φ is false are complements of each other, so it suffices to associate with φ only one set, say $[\![\varphi]\!]_M$. In a 3-valued logic, the set of states where φ is true does not determine the set of states where φ is false. Rather, we must consider three mutually exclusive and exhaustive sets: the set where φ is true, the set where φ is false, and the set where φ is undefined. As before, one of these is redundant, since it is the complement of the union of the other two. Note that if φ is a formula in the language of HMS, the set $[\![\varphi]\!]_M$ is either $\emptyset$ or an event of the form $B^\uparrow$, where $B \subseteq S_\alpha$. In the latter case, we associate with φ the pair of sets $(B^\uparrow, (S_\alpha - B)^\uparrow)$, i.e., $([\![\varphi]\!]_M, [\![\neg\varphi]\!]_M)$. In the former case, we must have $[\![\neg\varphi]\!]_M = S_\alpha^\uparrow$ for some α, and we associate with φ the pair $(\emptyset, S_\alpha^\uparrow)$. Thus, we are using the pair $(\emptyset, S_\alpha^\uparrow)$ instead of devising a new event $\emptyset^{S_\alpha}$ to represent $[\![\varphi]\!]_M$ in this case.[6]

[6]In a more recent version of their paper, HMS identify a nonempty event E with the pair (E, S), where, for $E = B^\uparrow$, S is the unique set S_α containing B. Then $\emptyset^S$ can be identified with $(\emptyset, S)$. While we also identify events with pairs of sets and $\emptyset^S$ with $(\emptyset, S)$, our identification is different from that of HMS, and extends more naturally to sets that are not events.

HMS use intersection of events to represent conjunction. It is not hard to see that the intersection of events is itself an event. The obvious way to represent disjunction is as the union of events, but the union of events is in general not an event. Thus, HMS define a disjunction operator using de Morgan's law: $E \vee E' = \neg(\neg E \cap \neg E')$. In our setting, where we use pairs of sets, we can also define operators $\sim$ and $\sqcap$ (intuitively, for negation and intersection) by taking $\sim(E, E') = (E', E)$ and

$$(E, E') \sqcap (F, F') = (E \cap F, (E \cap F') \cup (E' \cap F) \cup (E' \cap F')).$$

Although our definition of $\sqcap$ may not seem so intuitive, as the next result shows, $(E, E') \sqcap (F, F')$ is essentially equal to $(E \cap F, \neg(E \cap F))$. Moreover, our definition has the advantage of not using $\neg$, so it applies even if E and F are not events.

Lemma A.1: *If $(E \cup E') = S_\alpha^\uparrow$ and $(F \cup F') = S_\beta^\uparrow$, then*

$$(E \cap F') \cup (E' \cap F) \cup (E' \cap F') = \begin{cases} \neg(E \cap F), & \text{if } (E \cap F) \neq \emptyset \\ S_\gamma^\uparrow, & \text{if } (E \cap F) = \emptyset \text{ and } \gamma = \sup(\alpha, \beta). \end{cases}$$

Note that $\sup(\alpha, \beta)$ is well defined since Δ is a lattice.

Finally, HMS define an operator K_i corresponding to the possibility correspondence $\mathcal{K}_i$. They define $\mathsf{K}_i(E) = \{s : \mathcal{K}_i(s) \subseteq E\}$,[7] and show that $\mathsf{K}_i(E)$ is an event if E is. In our setting, we define

$$\mathsf{K}_i((E, E')) = (\{s : \mathcal{K}_i(s) \subseteq E\} \cap (E \cup E'), (E \cup E') - \{s : \mathcal{K}_i(s) \subseteq E\}).$$

Essentially, we are defining $\mathsf{K}_i((E, E')) = (\mathsf{K}_i(E), \neg\mathsf{K}_i(E))$. Intersecting with $E \cup E'$ is unnecessary in the HMS framework, since their conditions on frames guarantee that $\mathsf{K}_i(E) \subseteq E \cup E'$. If we think of (E, E') as $([\![\varphi]\!]_M, [\![\neg\varphi]\!]_M)$, then φ is defined on $E \cup E'$. The definitions above guarantee that $K_i\varphi$ is defined on the same set.

HMS define an awareness operator in the spirit of MR, by taking $\mathsf{A}_i(E)$ to be an abbreviation of $\mathsf{K}_i(E) \vee \mathsf{K}_i\neg\mathsf{K}_i(E)$. They then prove a number of properties of knowledge and awareness, such as $\mathsf{K}_i(E) \subseteq \mathsf{K}_i\mathsf{K}_i(E)$ and $\mathsf{A}_i(\neg E) = \mathsf{A}_i(E)$.

The semantics we have given for our logic matches that of the operators defined by HMS, in the sense of the following lemma.

Lemma A.2: *For all formulas $\varphi, \psi \in \mathcal{L}_n^{K,\hookrightarrow}(\Phi)$ and all HMS structures M.*

(a) $([\![\neg\varphi]\!]_M, [\![\neg\neg\varphi]\!]_M) = \sim([\![\varphi]\!]_M, [\![\neg\varphi]\!]_M))$

(b) $([\![\varphi \wedge \psi]\!]_M, [\![\neg(\varphi \wedge \psi)]\!]_M) = ([\![\varphi]\!]_M, [\![\neg\varphi]\!]_M) \sqcap ([\![\psi]\!]_M, [\![\neg\psi]\!]_M)$.

(c) $([\![K_i\varphi]\!]_M, [\![\neg K_i\varphi]\!]_M) = \mathsf{K}_i(([\![\varphi]\!]_M, [\![\neg\varphi]\!]_M))$

It is worth noting that Lemma A.2 applies even though, once we introduce the $\hookrightarrow$ operator, $[\![\varphi]\!]_M$ is not in general an event in the HMS sense. (For example, $[\![p \hookrightarrow q]\!]_M$ is not in general an event.)

[7] Actually, this is their definition only if $\{s : \mathcal{K}_i(s) \subseteq E\} \neq \emptyset$; otherwise, they take $\mathsf{K}_i(E) = \emptyset^{S_\alpha}$ if $E = B^\uparrow$ for some $B \subseteq S_\alpha$. We do not need a special definition if $\{s : \mathcal{K}_i(s) \subseteq E\} = \emptyset$ using our approach.

B An Axiomatization of $\mathcal{L}^{\hookrightarrow}(\Phi)$

Note that the formulas $\varphi = 0$, $\varphi = \frac{1}{2}$, and $\varphi = 1$ are 2-valued. More generally, we define a formula φ to be *2-valued* if $(\varphi = 0) \vee (\varphi = 1)$ is valid in all HMS structures. Because they obey the usual axioms of classical logic, 2-valued formulas play a key role in our axiomatization of $\mathcal{L}^{\hookrightarrow}(\Phi)$. We say that a formula is *definitely 2-valued* if it is in the smallest set containing $\top$ and all formulas of the form $\varphi = k$ which is closed under negation, conjunction, nonstandard implication, and K_i, so that if φ and ψ are definitely two-valued, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \hookrightarrow \psi$, and $K_i\varphi$. Let D_2 denote the set of definitely 2-valued formulas.

The following lemma is easy to prove.

Lemma B.1: *If φ is definitely 2-valued, then it is 2-valued.*

Let AX_3 consist of the following collection of axioms and inference rules:

- P0. $\top$.
- P1. $(\varphi \hookrightarrow \psi) \rightleftharpoons \neg(\varphi \wedge \neg\psi)$ if $\varphi, \psi \in D_2$.
- P2. $\varphi \hookrightarrow (\psi \hookrightarrow \varphi)$ if $\varphi, \psi \in D_2$.
- P3. $(\varphi \hookrightarrow (\psi \hookrightarrow \varphi')) \hookrightarrow ((\varphi \hookrightarrow \psi) \hookrightarrow (\varphi \hookrightarrow \varphi'))$ if $\varphi, \psi, \varphi' \in D_2$.
- P4. $(\varphi \hookrightarrow \psi) \hookrightarrow ((\varphi \hookrightarrow \neg\psi) \hookrightarrow \neg\varphi)$ if $\varphi, \psi \in D_2$.
- P5. $(\varphi \wedge \psi) = 1 \rightleftharpoons (\varphi = 1) \wedge (\psi = 1)$.
- P6. $(\varphi \wedge \psi) = 0 \rightleftharpoons (\varphi = 0 \wedge \neg(\psi = 1/2)) \vee (\neg(\varphi = 1/2) \wedge \psi = 0)$.
- P7. $\varphi = 1 \rightleftharpoons (\neg\varphi) = 0$.
- P8. $\varphi = 0 \rightleftharpoons (\neg\varphi) = 1$.
- P9. $(\varphi \hookrightarrow \psi) = 1 \rightleftharpoons ((\varphi = 0 \wedge \neg(\psi = 1/2)) \vee (\varphi = 1/2) \vee (\varphi = 1 \wedge \psi = 1))$.
- P10. $(\varphi \hookrightarrow \psi) = 0 \rightleftharpoons (\varphi = 1 \wedge \psi = 0)$.
- P11. $(\varphi = 0 \vee \varphi = 1/2 \vee \varphi = 1) \wedge (\neg(\varphi = i \wedge \varphi = j))$, for $i, j \in \{0, 1/2, 1\}$ and $i \neq j$.
- R1. From $\varphi = 1$ infer φ.
- MP′. From φ and $\varphi \hookrightarrow \psi$ infer ψ.

To show that a valid formula $\varphi \in \mathcal{L}^{\hookrightarrow}(\Phi)$ is provable in AX_3, we first prove that $\varphi = 1$ is provable in AX_3, and then apply R1 to infer φ. Axioms P5-P10 are basically a translation to formulas of the semantics for conjunction, negation and implication. They allow us to convert $\varphi = 1$ to a Boolean combination φ' of formulas of the form $p = k$, for $k \in \{0, 1, 2\}$, where p is a primitive proposition in Φ. It is well known that P0-P4 together with MP′ provide a complete axiomatization for classical 2-valued propositional logic with negation, conjunction, implication, and $\top$.[8] Therefore, any classical propositional tautology where the primitive propositions are formulas of the form $p = k$ follows from these axioms and inference rule. In particular, we can prove φ' using these axioms, using the fact (given by P11) that formulas have exactly one truth value. Thus, we can prove the following.

Theorem B.2: *AX_3 is a complete and sound axiomatization of $\mathcal{L}^{\hookrightarrow}(\Phi)$.*

Note that all the axioms of AX_3 are sound in classical logic (all formulas of the form $\varphi = 1/2$ are vacuously false in classical logic). Moreover, if add the axiom $\neg(\varphi = 1/2)$ to AX_3, we get a sound and complete axiomatization of classical propositional logic (although many axioms then become redundant).

[8] We remark that we included formulas of the form $K_i\varphi$ among the formulas that are definitely 2-valued. While such formulas are not relevant in the axiomatization of $\mathcal{L}^{\hookrightarrow}(\Phi)$, they do play a role when we consider the axiom Prop′ in $AX_n^{K,\hookrightarrow}$, which applies to instances in the language $\mathcal{L}_n^{K,\hookrightarrow}(\Phi)$ of valid formulas of $\mathcal{L}^{\hookrightarrow}(\Phi)$.