First-Order Classical Modal Logic: Applications in logics of knowledge and probability

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Abstract

The paper focuses on extending to the first order case the semantical program for modalities first introduced by Dana Scott and Richard Montague. We focus on the study of neighborhood frames with constant domains and we offer a series of new completeness results for salient classical systems of first order modal logic. Among other results we show that it is possible to prove strong completeness results for normal systems without the Barcan Formula (like $\mathbf{FOL} + \mathbf{K}$) in terms of neighborhood frames with *constant* domains. The first order models we present permit the study of many epistemic modalities recently proposed in computer science as well as the development of adequate models for monadic operators of high probability. We conclude by offering a general completeness result for the entire family of first order classical modal logics (encompassing both normal and non-normal systems).

1 Introduction

Dana Scott and Richard Montague (influenced by the classic result of McKinsey and Tarski [29]) proposed in 1970 (independently, in [36] and [31] respectively) a new semantic framework for the study of modalities, which today tends to be known as *neighborhood semantics*.

A **propositional neighborhood frame** is a pair $\langle W, N \rangle$, where W is a set of states, or worlds, and $N : W \to 2^{2^W}$ is a neighborhood function which associates a set of neighborhoods with each world. The tuple $\langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a neighborhood frame and V a valuation is a neighborhood model. A modal necessity operator is interpreted in this context as follows: $\mathcal{M}, w \models \Box\phi$ iff $(\phi)^{\mathcal{M}} \in N(w)$, where $(\phi)^{\mathcal{M}}$ is the truth-set corresponding to ϕ in the given model.

Without imposing specific restrictions on the neighborhood function it is clear that many important principles of normal or Kripkean modal logics will not hold in a neighborhood model. At the same time it is possible to show that there is a class of neighborhood models, the so-called *augmented* models (see the definition in the Appendix), which are elementary equivalent to the Kripkean models for normal modal systems of propositional modal logic. So, the program of neighborhood semantics has normally been considered as a generalization of Kripke semantics, which permits the study of classical systems that fail to be normal.

Early on (in 1971) Krister Segerbeg wrote an essay [38] presenting some basic results about neighborhood models and the modal axiom systems that correspond to them and later on Brian Chellas incorporated these and other salient results in part III of his textbook [11]. Following Chellas, we call axiom systems weaker than **K** *classical modal logics.* Nevertheless for more than 15 years or so after 1971, in the apparent absence of applications or in the absence of guiding intuitions concerning the role of neighborhoods, classical modal logics were studied mainly in function of their intrinsic mathematical interest. This situation has changed in important ways during the last 18 or so years. In fact, many of the normal axioms, like the additivity principle, establishing the distribution of the box over conjunction, have been found problematic in many applications. As a result many formalisms proposed to retain:

(M) $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

while abandoning:

(C) $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

Many recently explored, and independently motivated formalisms, have abandoned the full force of additivity while retaining monotony (M). Examples are Concurrent Propositional Dynamic Logic [21], Parikh's Game Logic [32], Pauly's Coalition Logic [33] and Alternating-Time Temporal Logic [1]. For instance, Pauly's coalitional logic [33] contains formulas of the form $[C]\phi$ intended to mean that the group of agents C have a joint strategy to ensure that ϕ is true (at the next stage of a game). Now, obviously, if a coalition has a strategy to make $\phi \wedge \psi$ true, that coalition also has a strategy to make ϕ true and a strategy to make ψ true (the same strategy can be used). However, if the coalition has a strategy to force ϕ and a strategy to force ψ, then the coalition may not necessarily have a strategy which results in both ϕ and ψ being true. For the strategy used to force ϕ to be true and the strategy used to force ψ to be true need not be the same or even compatible. Hence, Coalitional Logic accepts the axiom schem M but drops C.

Moreover recent research [5] has shown that a large family of Non-Adjunctive logics, previously studied only syntactically or via a variety of idiosyncratic extensions of Kripke semantics can be parametrically classified neatly as members of a hierarchy of monotonic classical logics, all of which admit clear and simple neighborhood models. A salient member of this family is the logic of monadic operators of high probability studied via neighborhood semantics in [25] and [4]. This is a clear case where the intended interpretation of neighborhoods is quite intuitive: the neighborhoods of a point are the propositions receiving probability higher than a fixed threshold. This will be discussed in more detail in Section 3.

More generally one can see the neighborhoods as having an epistemic role (as suggested in [4]), namely representing the knowledge, belief or certain other attitude of an agent at a certain point. This strategy permits the development of elegant and economic models of attitudes that can only be modeled alternatively via the abandonment of the axiom of foundation in set theory (or via co-algebras). In fact, many contemporary epistemic logics propose the use of 'composite' points encompassing both a description of the environment and a representation of the epistemic state of the agent. If this model is provided by specifying this epistemic state at a certain point as a set of (accepted) propositions which are components of the point itself we have an obvious circularity that can either be cured by abandoning the axiom of Foundation (see [7]) or by appealing to a co-algebraic representation. The use of neighborhoods permits the development of a simpler model of attitudes in these cases.

Unfortunately the recent interest in articulating applications for neighborhood semantics has not motivated yet the systematic study of first order classical logics and first order neighborhood models. One of us (Arló-Costa) presented in [4] preliminary results in this area showing that the role of the Barcan schemas in this context is quite different from the corresponding role of these schemas in the Kripkean case. In fact, the use of neighborhood semantics permits the development of models *with constant domains* where neither the Barcan (BF) nor the Converse Barcan formulas (CBF) are valid. Moreover [4] provides necessary and sufficient conditions for the validity of BF and CBF.

The recent foundational debates in the area of quantified modal logic oppose, on the one hand, the so called 'possibilists' who advocate the use of quantifiers ranging over a fixed domain of possible individuals, and on the other hand, the 'actualists' who prefer models where the assumption of the constancy of domains is abandoned. A salient feature of the standard first order models of modalities is that for those models the constancy of domains requires the validity of both the BF and the CBF (see [16] for a nice proof of this fact). Many philosophers have seen the possibilist approach as the only one tenable (see for example, [12], [28], [41]), and as a matter of fact the possibilist approach is the one that seems natural in many of the epistemic and computational applications that characterize the wave of recent research in modal logic (see, for example, the brief section devoted to this issue in [15]). Nevertheless, while the possibilist approach seems reasonable on its own, the logical systems that adopt the Barcan Formulas and predicate logic rules for the quantifiers might be seen as too strong for many applications. The problem is that in a Kripkean framework one cannot have one without the other. This has motivated some authors to adopt more radical approaches where quantifiers range over individual concepts (functions from possible worlds to the domain of objects). The approach provided by Garson [18] in particular is quite ingenious although it seems limited to first order extensions of **K** (and the use of individual concepts does not seem immediately motivated in some simple applications considered here, like the logic of high probability).

Notice, for example, that (as indicated in [4]) if the box operator is understood as a monadic operator of high probability the BF can be interpreted as saying that if ticket of a lottery is a loser then all tickets are losers. While the CBF seems to make sense as a constraint on an operator of high probability the BF seems unreasonably strong. At the same time nothing indicates that a possibilitic interpretation of the quantifiers should be abandoned for representing a monadic operator of high probability. On the contrary the possibilistic approach seems rather natural for this application. It is therefore comforting that one can easily develop first order neighborhood semantics with constant domains where this asymmetry is neatly captured (i.e. where the CBF is validated but the BF is not). In particular we argue that the system $\mathbf{FOL}+\mathbf{E}+CBF$ is adequate for representing first order monadic operators of high probability - **E** is the weakest system of classical modal logic.

A general completeness result for all systems of first order classical modal logic is presented in terms of general neighborhood frames (which we call *cylindrical* general frames). This result (immune to incompleteness results that plague both the Kripkean and the Scottian semantics) has the important consequence of making the use of varying domains optional but not mandatory in the study of first order modal logic without the Barcan schema. There are also interesting connections between our results and topological sematics for modal logic [9], [6].

We utilize various operators of high probability and expectation in order to give concrete examples of first order modalities, both normal and non-normal, that can be elegantly modeled via first order neighborhoods but that lack model-theoretical characterizations in terms of relational models with constant domains. Other interesting examples abound in epistemic logic, game logic and conditional logic. Applications in game theory are also possible (related to characterization of the hierarchy of high probability operators used in Harsanyi's theory of types).

2 Classical First-Order Modal Logic: Syntax and Semantics

The reader is referred to the Appendix for information about classical propositional modal logic. The language of first order modal logic is defined as follows. Let $\mathcal{V}$ be a countable collection of individual variables. For each natural number $n \geq 1$, there is a (countable) set of n-place predicate symbols. These will be denoted by $F, G, \ldots$. In general, we will not write the arity of a predicate F. A formula of *first order modal logic* will have the following syntactic form

$$\phi := F(x_1, \ldots, x_n) \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \forall x\phi$$

Let $\mathcal{L}_1$ be the set of well-formed first order modal formulas. The other standard Boolean connectives, the diamond modal operator and the existential quantifier are defined as usual. The usual rules about free variables apply. We write $\phi(x)$ when x (possibly) occurs free in ϕ. Denote by $\phi[y/x]$, ϕ in which free variable x is replaced with free variable y. The following axioms are taken from [24]. Let **S** be any classical propositional modal logic, let **FOL** + **S** be the set of formulas closed under the following rules and axiom schemes:

S All axiom schemes and rules from **S**.

$\forall$ $\forall x\phi(x) \rightarrow \phi[y/x]$ is an axiom scheme. [1]

Gen $\dfrac{\phi \rightarrow \psi}{\phi \rightarrow \forall x\psi}$, where x is not free in ϕ.

For example, **FOL** + **E** contains the axiom scheme $PC, E, \forall$ and the rules Gen, MP, RE.[2] Given any classical propositional modal logic **S**, we write $\vdash_{\mathbf{FOL+S}} \phi$ if $\phi \in$ **FOL** + **S** (equivalently ϕ can be derived using the above axiom schemes and rules).

Notice that in the above axiom system there is no essential interaction between the modal operators and the first-order quantifiers. Two of the most widely discussed axiom schemes that allow interaction between the modal operators and the first-order quantifiers are the so-called Barcan formula and the converse Barcan formula.

Definition 2.1 *Any formula of the form*

$$\forall x\Box\phi(x) \rightarrow \Box\forall x\phi(x)$$

will be called a **Barcan formula** *(BF). The* **converse Barcan formula** *(CBF) will be any formula of the form*

$$\Box\forall x\phi(x) \rightarrow \forall x\Box\phi(x)$$

Technically, the Barcan and converse Barcan formulas aree schemes not formulas, but we will follow standard terminology. For simplicity, we will write **S** + BF for the logic that includes all axiom schemes and rules of **FOL** + **S** plus the Barcan formula BF. Similarly for **S** + CBF.

Definition 2.2 *A* **constant domain neighborhood frame** *for classical first-order modal logic is a tuple $\langle W, N, D\rangle$, where W is a set of possible worlds, N is a neighborhood function and D is any non-empty set, called the* **domain***.*

Definition 2.3 *A* **constant domain neighborhood model** *based on a frame $\mathcal{F} = \langle W, N, D\rangle$ is a tuple $\mathcal{M} = \langle W, N, D, I\rangle$, where I is a classical first-order interpretation function. Formally, for each n-ary predicate symbol F, $I(F, w) \subseteq D^n$.*

[1]According to the notation used in [24], which we are following here, this axiom follows from two additional principles called the principles of *replacement* and *agreement*. See [24] page 241. These principles guarantee that y is free for x occurring in $\phi(x)$.

[2]See Appendix A for details about sub-Kripkean classical modal logics, including the axiomatization of **E**, etc.

An **assignment** is any function $\sigma : \mathcal{V} \to D$. An assignment σ' is said to be an x**-variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$. Truth is defined at a state relative to an assignment. Let $\mathcal{M} = \langle W, N, D, I\rangle$ be any constant domain neighborhood model and σ any assignment.

1. $\mathcal{M}, w \models_\sigma F(x_1, \dots, x_n)$ iff $\langle \sigma(x_1), \dots, \sigma(x_n)\rangle \in I(F, w)$ for each n-place predicate symbol F.
2. $\mathcal{M}, w \models_\sigma \neg\phi$ iff $\mathcal{M}, w \not\models_\sigma \phi$
3. $\mathcal{M}, w \models_\sigma \phi \wedge \psi$ iff $\mathcal{M}, w \models_\sigma \phi$ and $\mathcal{M}, w \models_\sigma \psi$
4. $\mathcal{M}, w \models_\sigma \Box\phi$ iff $(\phi)^{\mathcal{M},\sigma} \in N(w)$
5. $\mathcal{M}, w \models_\sigma \forall x\phi(x)$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} \phi(x)$

where $(\phi)^{\mathcal{M},\sigma} \subseteq W$ is the set of states $w \in W$ such that $\mathcal{M}, w \models_\sigma \phi$.

The following observations are well-known and easily checked (see [24] page 245).

Observation 2.4 ([24]) *The converse Barcan formula is provable in the logic* $\mathbf{FOL} + \mathbf{K}$

Observation 2.5 ([24]) *The Barcan formula is valid in all first order Kripke models with constant domains.*

Since the weakest propositional modal logic sound and complete for all Kripke frames is $\mathbf{K}$, we will focus on the logic $\mathbf{FOL} + \mathbf{K}$. Given Observation 2.5, if we want a completeness theorem with respect to all constant domain Kripke structures, we need to consider the logic $\mathbf{K} + BF$. Soundness is shown via the following theorem (Corollary 13.3, page 249 in [24]). Given any Kripke frame $\mathcal{F}$, we say that $\mathcal{F}$ is a frame for a logic $\mathbf{S}$ iff every theorem of $\mathbf{S}$ is valid on $\mathcal{F}$.

Theorem 2.6 ([24] page 249) *Let* $\mathbf{S}$ *be any propositional normal modal logic, then a Kripke frame* $\mathcal{F}$ *is a frame for* $\mathbf{S}$ *iff* $\mathcal{F}$ *is a frame for* $\mathbf{S} + BF$.

The proof of the only if direction is a straightforward induction on a derivation in the logic $\mathbf{S} + BF$. As for the if direction, the main idea is that given $\mathcal{F}$ which is not a frame for $\mathbf{S}$, one can construct a frame $\mathcal{F}^*$ that is not a frame for $\mathbf{S} + BF$.

Essentially this theorem shows that the notion of a frame for first order Kripke models is *independent* of the domain D. That is, the proof of the above theorem goes through *whatever* D *may be.* Let Λ of formulas of first order modal logic.

Definition 2.7 *A set* Λ *has the* $\forall-$**property**[3] *iff for each formula* ϕ *and each variable* x*, there is some variable* y*, called the witness, such that* $\phi[y/x] \to \forall x\phi(x) \in \Lambda$.

The proof of the following Lindenbaum-like Lemma is a straightforward and can be found in [24] page 258

Lemma 2.8 *If* X *is a consistent set of formulas of* $\mathcal{L}_1$*, then there is a consistent set of formulas* Y *of* $\mathcal{L}_1^+$ *with the* $\forall$*-property such that* $X \subseteq Y$*, where* $\mathcal{L}_1^+$ *is the language* $\mathcal{L}_1$ *with countably many* new *variables.*

[3]The terminology follows the one used in [24]. The property is also known as Henkin's property.

3 First-order modal logics of high probability

In this section we discuss a few examples of first-order modal logics of high probability. These examples provide a motivation for a number of systems studied in the next section. In particular, we focus on **FOL + EMN** and **FOL + K**. The first example describes a class of neighborhood frames which are axiomatized by **FOL + EMN** (Theorem 5.5). The second examples describes a class of neighborhood frames axiomatized by **FOL + K** (Theorem 5.10).

Example 1: Qualitative probability defined over rich languages The system **EMN** and first order extensions of it seem to play a central role in characterizing monadic operators of high probability. These operators have been studied both in [25] and in [4]. Roughly the idea goes as follows: let W be a set of states and Σ_W a σ-algebra generated by W. Let $P : \Sigma_W \to [0,1]$ be a probability measure and $t \in (0.5, 1)$. Let $\mathcal{H}_t \subseteq \Sigma_W$ be the set of events with "high" probability with respect to t, that is $\mathcal{H}_t = \{X \mid P(X) > t\}$. It is easy to see that $\mathcal{H}_t$ is closed under superset (**M**) and contains the unit (**N**). A similar construction is offered in [25], where the authors claim that the resulting propositional logic is **EMN**,[4]

The accounts of monadic operators of high probability presented in [25] and [4] can be generalized by appealing to the tools presented here. We will sketch in this example how this can be done by appealing to the account offered in [17], which offers a model of high probability operators for a Kolmogorovian notion of probability defined over a first order language.

Let L_0 be a first order language for arithmetic. So L_0 has names for the members of N, aside from symbols for addition and multiplications, variables that take values on N and quantifiers, etc. Notice that this language is richer than the one used above. For example, it has constants, which in this case are numerals $n_1, \ldots$. We use 'n' ambiguously for natural numbers and their numerals. Let in addition L be an extension of L_0 containing a finite amount of atomic formulas of the form $R(t_0, \ldots, t_k)$ where t_i is either a variable or a numeral. Let Pr be a nonnegative real-valued function defined for the sentences of L and such that the following conditions hold:

1 If $\models \psi \leftrightarrow \phi$ then $Pr(\psi) = Pr(\phi)$

2 If $\models \psi$, then $Pr(\psi) = 1$.

3 If $\models \neg(\psi \wedge \phi)$, then $Pr(\psi \vee \phi) = Pr(\psi) + Pr(\phi)$

4 $Pr(\exists x \phi(x)) = Sup\{Pr(\phi(n_1) \vee \ldots \vee \phi(n_k)) : n_1 \ldots n_k \in N,\ k = 1, 2, \ldots\}$

Condition (4) is the substantive condition, which in [37] is called ' Gaifman's condition'. It is clear that we can use this notion of probability in order to define a modality 'ψ is judged as highly probable by individual i' modulo a threshold t. So, for example, for (a finite stock of) atomic formulas we will have

$\mathcal{M}, w \models_\sigma \Box R(t_1, \ldots, t_n)$ if and only if $|R(t_1, \ldots, t_n)|^{\mathcal{M},\sigma} \in N(w)$ if and only if $Pr_w(R(\sigma(t_1), \ldots, \sigma(t_n)) > t$.

And in general neighborhoods contain the propositions expressed by sentences of L to which an agent of reference assigns high probability. Of course, other interesting modalities, like 'sequence s

[4]The model offered in [25] differs from the one sketched here in various important manners. First it attributes probability to sentences, not to events in field. Second it works with a notion of primitive conditional probability which is finitely additive. Third probability in Kyburg and Teng's model is not personal probability but objective chance, which is interval-valued.

is random' or 'ψ is judged as more probable than ϕ', etc. can also be defined. In view of the previous results we conjecture that the logic that thus arises should be an extension of the *non-nested fragment* of **FOL + EMN** – we refer to the logic encoding the valid formulas determined by the aforementioned high probability neighborhoods. Of course, BF should fail in this case, given that it instantiates cases of what is usually known as the 'lottery paradox' •

Example 2: Finitely additive conditional probability When neighborhoods encode qualitative expectations for finitely additive measures (in countable spaces) they form non-augmented filters validating **FOL + K** but not *BF*.

Some distinguished decision theorists (like Bruno De Finetti and Leonard J. Savage) as well as some philosophers (Isaac Levi) have advocated the use of finitely additive conditional probability in the decision sciences. Lester E. Dubins offers an axiomatic characterization of finitely additive conditional probability in [14]. As De Finetti suggested in [13] it is possible to extract a *superiority* ordering from finitely additive probability, which has, in turn, been used more recently in order to define full belief and expectations from primitively given conditional probability in a paradox-free manner ([40], [3]). Moreover finitely additive conditional probability has also been used in order to define non-monotonic notions of consequence ([3], [20]). We will show here that the resulting modalities can be represented in salient cases by neighborhood frames forming non-augmented filters.

We will proceed as follows: we will first define a qualitative structure in neighborhoods (basically a non-augmented filter). Then we will define a conditional measure satisfying Dubins' axioms by utilizing this qualitative structure. Finally we will show that the structure in question offers a characterization of belief and expectation corresponding to the measure.

Let the space be the (power set of the) positive integers and consider E = {2n: n = 1, ...} be the even integers in the space, and let O = {2n-1: n = 1, ...} be the odd integers in the space. Let $P(i) = \{1/2^n$: if $i = 2n\}$ and let $P(i) = 0$, otherwise. So the unconditional P is countably additive, whose support is the even integers E. But $P(.|O)$ might be uniform on the odd integers. This can be reflected by the fact that there is a *core system* over O defined as follows: Let the outermost core $C_1 = O$. Then $C_2 = \{1\}^c$, $C_3 = \{1, 3\}^c$, etc. For any n: 1, ...; define a rank system r_n for C_n as follows:

(ranks) $r_n = \{w \in \Omega$: $w \in C_n$ - C_{n+1} }

Notice that each rank contains exactly one odd number with $r_1 = \{1\}$. Now we can define conditional probability as follows. If there is a largest integer i such that $A \cap C_i \neq \emptyset$, define $Q(B|A)$ (for $A, B \subseteq O$) as: p_i(A ∩ B) / $p_i(A)$. Otherwise set $Q(B|A)$ to 1 if there is r_i such that $A \cap C_i \neq \emptyset$, and $A \cap C_i \subseteq B$ and $Q(B^c|A)$ to 0 for Bs satisfying the given conditions. For the remaining infinite sets such that both B and B^c are infinite, arbitrarily set one of them to 1 and the complement to 0. Finally set $Q(B|A)$ to 0 for every other event in the space. According to this definition each co-finite set in O has measure 1 (because it is entailed by at least a core) and each number in O carries zero probability. Of two infinite but not co-finite sets, say S = {1, 5, 9,...} and H = {3, 7, 11, ...}, we can assign 1 to the set of lower rank (so S carries measure 1 and H zero).

We shall now formally introduce the notion of *probability core*. We follow here ideas presented in [3], which, in turn, slightly modify the schema first proposed in [40]. A notion that plays an important role in those works is the notion of *normality*. The basic idea is that an event A is normal for Q as long as $Q(\emptyset|A) = 0$. Conditioning on abnormal events would lead to incoherence.

A probability *core* for Q is an event K which is normal and satisfies the *strong superiority* *condition* (SSC) i.e. if A is a nonempty subset of K and B is disjoint from K, then $Q(B|A \cup B) = 0$ (and so $Q(A|A \cup B) = 1$). Thus any non-empty subset of K is more "believable" than any set disjoint from K.

Now it is easy to see that the system of cores C_n constructed above constitutes a system of cores for Q as defined in the previous paragraphs. Cores can be used in order to characterize *qualitative expectations* relative to Q: an event E is expected relative to Q as long as it is entailed by some core for Q. So, given Q we can construct all neighborhoods of a frame as the set of expectations for Q. A binary modality can also be intuitively characterized, namely conditional expectation: relative to any finite subset of O its largest element will always be expected. It is easy to see that these neighborhoods form non-augmented filters and that the first order logic of qualitative expectations should at least obey the axioms of the non-nested fragment of $\mathbf{FOL} + \mathbf{K}$.

Expectations have been utilized in [3] in order to define non-monotonic consequence: B is non-monotonically entailed by A relative to Q if and only if B is expected relative to $Q(.|A)$.[5] An important argument in [3] shows that in infinite spaces and for logically infinite languages (equipped with at least a denumerable set of atoms) the definition of non-monotonic consequence sketched above obeys standard axioms of *rational consequence* proposed by Lehman and Magidor [26] only if the underlying measure is not countable additive, so the filter structure presented above is essentially needed in order to characterize probabilistically both qualitative expectation and conditionals (relative to a conditional measure Q) •

4 More examples: Non-adjunctive logics and logics of knowledge and time

The logics for monadic operators of high probability are just an example of logics which fail to obey axiom (C). There is a larger family of Non-Adjunctive logics among which we can find some of the first paraconsistent logics. Schotch and Jennings have offered in [35] one of the standard contemporary systems of non-adjunctive inference. One of the central ideas considered by Schotch and Jennings' is their proposal for measuring the coherence of a set of sentences. Their *coherence function* c is a function having as its domain the set of all finite sets of sentences and as a codomain the set $Nat \cup \{w\}$, where Nat is the set of natural numbers.

Definition 4.1 *For* **false** $\notin \Gamma$, $c(\Gamma) = m$ *if and only if* m *is the least integer such that there are sets*

$a_1, ..., a_m$, *with* $a_i \nvdash$ **false** *(*$1 \leq i \leq m$*)*

and $\cup_{i=1}^{m} a_i = \Gamma$

where $\vdash$ *is the classical notion of consequence and where* $c(\Gamma) = w$ *by convention if* **false** $\in \Gamma$

Now we can define a notion of derivability in terms of this notion of levels of coherence. The *forcing relation* $[\vdash$ is characterized as a relation between finite sets of sentences and sentences and defined as follows:

[5] Other notions of expectation proposed in the literature, like the one presented in [27], do not seem to exhibit the logical structure encoded in $\mathbf{FOL} + \mathbf{K}$; but they can also be analyzed with the tools offered here.

Definition 4.2 *For $c(\Gamma) = n(w)$, Γ $[\vdash A$ if and only if for every n-fold (w-fold) decomposition $a_1, ..., a_n$, of Γ, there is some i such that $a_i \vdash A (1 \leq i \leq n(w))$.*

The forcing relation obeys structural rules, which we do not present here for the sake of brevity. In addition we have the usual rules for introducing and eliminating connectives, with the notable exception that the rule for introducing conjunction only holds for sets Γ, such that $c(\Gamma) = 1$.

A modality that Schotch and Jennings' derive from this syntax can be neatly characterized in terms of neighborhood models of the sort considered here (this was done by one of us in [5]). This approach can be extended to provide a unified semantics for a large set of Non-Adjunctive logics. Moreover the models presented here show how to extend these modalities to the first order case.

4.1 Possible worlds in epistemic semantics

Moshe Vardi considered in [39] the use of classical modal systems in order to represent both failures of logical omniscience and high probability operators. He also stated in passing that the proposal could be used in order to circumvent the lottery paradox. Nevertheless, after considering the use of classical modal systems, Vardi discarded them without exploring their logical power. Vardi gave two reasons for not utilizing systems of classical modalities (which he dubs *intensional logic* following Montague's terminology). The central reason is that this approach 'leaves the notion of a possible world as a primitive notion [...]. While this might be seen as an advantage by the logician whose interest is in epistemic logic, it is a disadvantage for the "user" of epistemic logic whose interest is mostly in using the framework to model belief states (page 297)." Vardi proceeds instead to establish that: "a world consists of a truth assignment to the atomic propositions and a collection of sets of worlds. This is, of course, a circular definition...". Barwise and Moss [7] showed how to make this strategy coherent by abandoning the axiom of foundation in set theory. Since then this type of models has been used in developing epistemic foundations for solution concepts in game theory.

There is yet a different way of facing this problem [15]. The idea is to assume that $W \subseteq O \times S_1 \times ... \times S_n$ where O is the set of *objective* states and S_i is a set of *subjective* states for agent i. Therefore worlds have the form $(o, s_1, ..., s_n)$. In multi-agent systems o is called the *environment state* and each s_i is called a *local* state for the agent in question.

Halpern [22] characterizes an agent's subjective state s_i by saying that it represents 'i's perception of the world and everything else about the agent's makeup that determines the agent's reports'. It is unclear whether these subjective states can be fully characterized propositionally. Models of the sort we are reviewing here bypass this problem by taking not only the 'subjective' states in each S_i, but also the 'objective' states in O as primitives.

Some of the arguments presented here indicate that there might be good reasons for utilizing the strategy discarded by Vardi also for applications concerned to representing knowledge in multi-agent systems. The idea is to return to a view where worlds are primitives and associate them with neighborhoods containing propositional representations of the local states of the agents and the environment. This strategy has the advantage of being more comprehensive (allowing for natural and simple models of notions of likelihood) and of having very interesting first order extensions. Among other things we can provide general completeness results both for all normal and non-normal first order modal logics, with and without the Barcan schemas, in models with constant domains.

5 Completeness results

In this section we discuss the completeness of various classical systems of first-order modal logic. We start by defining the smallest canonical model for classical first-order modal logic. Let Λ be any first-order classical modal logic. Define $\mathcal{M}_\Lambda = \langle W_\Lambda, N_\Lambda, D_\Lambda, I_\Lambda \rangle$ as follows. Let $MAX_\Lambda(\Gamma)$ indicate that the set Γ is a Λ-maximally consistent set of formulas of $\mathcal{L}_1^+$.

$$W_\Lambda = \{\Gamma \mid MAX_\Lambda(\Gamma) \text{ and } \Gamma \text{ has the } \forall\text{-property}\}$$

$$X \in N_\Lambda(\Gamma) \text{ iff for some } \Box\phi \in \Gamma,\ X = \{\Delta \mid \Delta \in W_\Lambda, \phi \in \Delta\}$$

$$D_\Lambda = \mathcal{V}^+$$

$$\langle x_1, \ldots, x_n \rangle \in I_\Lambda(\phi, \Gamma) \text{ iff } \phi(x_1, \ldots, x_n) \in \Gamma$$

$$\text{For every variable } x \in \mathcal{V}^+, \sigma(x) = x$$

where $\mathcal{V}^+$ is the extended set of variables used in Lemma 3.8. The definition of the neighborhood function N_Λ essentially says that a set of states of the canonical model is necessary at a world Γ precisely when Γ *claims that it should.* For any formula $\phi \in \mathcal{L}_1$, let $|\phi|_\Lambda$ be the proof set of ϕ is the logic Λ, that is,

$$|\phi|_\Lambda = \{\Gamma \mid \Gamma \in W_\Lambda \text{ and } \phi \in \Gamma\}$$

The fact that N_Λ is a well-defined functions follows from the fact that Λ contains the rule *RE*.

Definition 5.1 *Let $\mathcal{M} = \langle W, N, D, I \rangle$ be any first-order constant domain neighborhood model. $\mathcal{M}$ is said to a be a* **canonical for** *a first-order classical modal axiom system Λ provided $W = W_\Lambda$, $D = D_\Lambda$, $I = I_\Lambda$ and*

$$|\phi|_\Lambda \in N(\Gamma) \textit{ iff } \Box\phi \in \Gamma$$

Thus the model $\mathcal{M}_\Lambda$ is the smallest canonical model for a logic Λ. It is shown in Chellas ([11]) that if $\mathcal{M} = \langle W, N, D, I \rangle$ is a canonical model, then so is $\mathcal{M}' = \langle W, N', D, I \rangle$,where for each $\Gamma \in W$, $N'(\Gamma) = N(\Gamma) \cup \{X \subseteq W \mid X \neq |\phi|_\Lambda \text{ for any } \phi \in \mathcal{L}_1\}$. That is N' is N with all of the non-proof sets.

Lemma 5.2 (Truth Lemma) *For each $\Gamma \in W_\Lambda$ and formula $\phi \in \mathcal{L}_1$,*

$$\phi \in \Gamma \textit{ iff } \mathcal{M}_\Lambda, \Gamma \models_\sigma \phi$$

Proof The proof is by induction on ϕ. The base case and propositional connectives are as usual. The quantifier case is exactly as in the Kripke model case. We need only check the modal cases. The proof proceeds easily by construction of N_Λ and definition of truth: $\Box\phi \in \Gamma$ iff (by construction) $|\phi| \in N_\Lambda(\Gamma)$ iff (by definition of truth) $\mathcal{M}_\Lambda, \Gamma \models_\sigma \Box\phi$.

Notice that in the above proof, as opposed to the analogous result for Kripke models, the Barcan formula is not needed. The following corollary follows from the truth Lemma via a standard argument.

Theorem 5.3 *For any canonical model $\mathcal{M}$ for a classical first-order modal logic Λ, ϕ is valid in the canonical model $\mathcal{M}$ iff $\vdash_\Lambda \phi$*

Corollary 5.4 *The class of all first-order neighborhood constant domain frames is sound and complete for* **FOL** + **E**.[6]

Notice that the in the canonical model $\mathcal{M}_\Lambda$ constructed above, for any state Γ, the set $\mathcal{N}_\Lambda(\Gamma)$ contains only proof sets, i.e., sets of the form $\{\Delta \mid \phi \in \Delta\}$ for some formula $\phi \in \mathcal{L}_1$. For this reason, even if Λ contains the M axiom scheme, N_Λ may not be supplemented. Essentially the reason is if $X \in N_\Lambda(\Gamma)$, and $X \subseteq Y$, Y may not be a proof set, so cannot possibly be in $N_\Lambda(\Gamma)$. However, it can be shown that the supplementation $\mathcal{M}_\Lambda^+$ is a canonical for **FOL** + **EM** (by adapting to the first order case the proof offered in [11] page 257).

Theorem 5.5 **FOL** + **EMN** *is sound and complete with respect to the class of non-trivial supplemented first-order neighborhood frames that contain the unit.*

Definition 5.6 *A classical system of first order modal logic Λ is* canonical *if and only if the frame of at least one of its canonical models is a frame for Λ.*

Now we can establish a result showing that **FOL** + **E** + CBF is not canonical (in the strong sense we just defined). Not only the frame of the smallest canonical for **FOL** + **E** + CBF is not a frame for **FOL** + **E** + CBF, but actually this applies to any other canonical frame including non-proof sets as well.

Observation 5.7 **FOL** + **E** + CBF *is not canonical.*

Proof Assume by contradiction that **FOL** + **E** + CBF is canonical. If this were so, given that (by construction) the frame of any canonical for **FOL** + **E** + CBF is non-trivial (the cardinality of the domain is strictly greater than 1), then there is a canonical model of **FOL** + **E** + CBF whose frame is a frame for **FOL** + **E** + CBF.

But then the frame in question should be supplemented. And if this were the case then every first order instance of **M** should be valid in the canonical model even when some of these instances are non-theorems of **FOL** + **E** + CBF. Contradiction.

It is easy to see that the previous argument can be extended to show that **FOL** + **E** + CBF is not sound and complete with respect to any non-trivial class of frames (i.e. a class where each frame is non-trivial). But the system is not essentially incomplete. It is not difficult to see that if we take the class of non-trivial supplemented first-order neighborhood frames and we add to it a trivial and non-supplemented frame this class fully characterizes **FOL** + **E** + CBF. This is so given that CBF continues to be valid with respect to the widened class, but the addition of the trivial frame guarantees that **M** is no longer valid in the widened class of frames.

Observation 5.8 *The augmentation of the smallest canonical model for* **FOL** + **K** *is not a canonical model for* **FOL** + **K**. *In fact, the closure under infinite intersection of the minimal canonical model for* **FOL** + **K** *is not a canonical model for* **FOL** + **K**.

Lemma 5.9 *The augmentation of the smallest canonical model for* **FOL** + **K** + BF *is a canonical for* **FOL** + **K** + BF.

[6]See the appendix for the definition of various systems of propositional classical logic, including **E**.

Theorem 5.10 **FOL + K** *is sound and complete with respect to the class of filters.*

Theorem 5.11 **FOL + K** + BF *is sound and complete with respect to the class of augmented first-order neighborhood frames.*

Completeness proofs for normal first-order modal logics have beens studies by a number of authors. In particular soundness and completeness of first-order **S4** was first shown by Rasiowa and Sikorski [34]. See [16] for a complete discussion.

6 Cylindrical Frames

It was shown first by M. Gerson [19] that the neighborhood semantics (even at the propositional level) suffers from a certain important inadequacy. The problem is that there are logics which are not complete with respect to *any* class of neighborhood frames. These logics are also incomplete with respect to relational semantics a la Kripke, so Gerson's result shows that neighborhood semantics inherits other types of important inadequacies of the Kripkean program. In addition it is also well known that the first order logics strictly between $S4.4$ and $S5$ *without* the Barcan Formula are also incomplete (in relational semantics). We conjecture that the latter incompleteness can be removed by using first order neighborhood frames. But the first kind of incompleteness cannot be eradicated by utilizing first order neighborhood frames.

The solution for this kind of incompleteness in relational semantics consists in adopting the so-called *general frames* constituted by a frame together with a restricted but suitably well-behaved set of *admissible valuations* – see [10] for a textbook presentation of general frames.[7] We show below that a similar solution is feasible for first order neighborhoods via the adoption of general first order neighborhood frames which we will call *cylindrical general frames*.[8]

Let $\mathcal{F} = \langle W, N, D\rangle$ if a first-order neighborhood frame with constant domain. The neighborhood function induces a function $N_\Box : 2^W \to 2^W$ defined as follows. Let $X \subseteq W$, then $N_\Box(X) = \{w \in W \mid X \in N(w)\}$. Intuitively, $N_\Box(X)$ is the set of states where the proposition X is necessary.

Let $\langle W, N, D\rangle$ be a first-order neighborhood frame with constant domain. Let ${}^\omega D$ denote be the set of all functions from ω to D. For $i \in \omega$, $s \in {}^\omega D$ and $d \in D$, let s^i_d denote the function which is exactly the same as s except for the ith component which is assigned d. For $s, s' \in {}^\omega D$, we say that s and s' are i-equivalent if $s' = s^i_d$ for some $d \in D$. When convenient, we will think of a function $s \in {}^\omega D$ as an infinite sequence of elements from D. Intuitively, these sequences represent an assignment $\sigma : \mathcal{V} \to D$. To make this representation concrete, we need to fix an ordering on the set of variables $\mathcal{V}$, i.e., assume that $\mathcal{V} = \{v_1, \ldots, v_n, \ldots\}$. We fix this ordering on the set of variables for the rest of this section. Then we can be more precise about the correspondence between sequences and assignments. For any assignment, $\sigma : \mathcal{V} \to D$, there is a unique sequence, denoted by s_σ, such that $s_\sigma(i) = \sigma(v_i)$.

Similarly, for each sequence $s \in {}^\omega D$, there is a unique substitution, denoted by σ_s, such that $\sigma_s(v_i) = s_i$. Obviously, we have that $\sigma \sim_{v_i} \sigma'$ implies $s_\sigma \sim_i s_{\sigma'}$ (and vice versa).

Consider a function $f : {}^\omega D \to 2^W$. Given a sequence $s \in {}^\omega D$, we think of $f(s)$ as a proposition (set of states) expressed by a relation where its finite set of free variables $v_{i_1}, \ldots, v_{i_k}$ are replaced by $s_{i_1}, \ldots, s_{i_k}$ and where $i_1, \ldots, i_k$ are the relevant coordinates[9].

[7] The terminology 'general frames' can be traced back to van Benthem's paper [8]. More complete historical references can be found in footnote 6 in chapter 9 of [24].

[8] The term 'cylindrical' is motivated by the work or Tarski on cylindrical algebras, which, in turn, motivates our algebraic treatment of first order relations – for an up to date presentation of cylindrical set algebras see [30].

[9] Recall that we have fixed an ordering of the set of variables $\mathcal{V}$.

Intuitively the function f should be thought of as representing the equivalence class of formulas logically equivalent to a relation $\phi(v_{i_1}, \dots, v_{i_n})$, in the sense that for each substitution σ, $f(s_\sigma) = (\phi(v_{i_1}, \dots, v_{i_n}))^{\mathcal{M},\sigma}$. Given a first-order neighborhood frame with constant domain $\mathcal{F} = \langle W, N, D \rangle$, we say that a collection of functions $\{f_i\}_{i \in \omega}$ is **appropriate for** $\mathcal{F}$ if,

1. For each $i \in \omega$, there is a $j \in \omega$ such that for all $s \in {}^{\omega}\mathcal{D}$, $f_j(s) = W - f_i(s)$. We will denote f_j by $\neg f_i$.
2. For each $i, j \in \omega$, there is a $k \in \omega$ such that for all $s \in {}^{\omega}\mathcal{D}$, $f_k(s) = f_i(s) \cap f_j(s)$. We will denote f_k by $f_i \wedge f_j$.
3. For each $i \in \omega$, there is a $j \in \omega$ such that for all $s \in {}^{\omega}\mathcal{D}$, $f_j(s) = N_\Box(f_i(s))$. We will denote f_j by $\Box f_i$.

Definition 6.1 *$\langle W, N, D, A, \{f_i\}_{i\in\omega}\rangle$ is a* **general cylindrical neighborhood frame with constant domain** *if $\langle W, N, D \rangle$ is a first-order neighborhood frame with constant domain, $A \subseteq 2^W$ is a collection of sets closed under complement, finite intersection and the $N_\Box$ operator and $\{f_i\}_{i\in\omega}$ is a countable set of functions $f : {}^{\omega}\mathcal{D} \to 2^W$ appropriate for $\langle W, N, D \rangle$, where for each i and each $s \in {}^{\omega}\mathcal{D}$, $f_i(s) \in A$ and each f_i satisfies the following condition:*

$$\text{(C)} \qquad \text{For each } v_i \in \mathcal{V}, \bigcap_{s' \sim_i s} f(s') \in A$$

An interpretation I is **A-admissible for** $\{f_i\}_{i\in\omega}$ in a general first-order neighborhood frame $\mathcal{F} = \langle W, N, D, A, \{f_i\}_{i\in\omega}\rangle$ provided for each n-ary atomic formula F and each assignment σ,

1. $((F(x_1, \dots, x_n))^{\mathcal{M},\sigma} \in A$, where $\mathcal{M}$ is the first-order neighborhood model based on $\mathcal{F}$ with interpretation I, and
2. For each n-ary atomic formula F, there is a function f_i representing it such that for each $s \in {}^{\omega}\mathcal{D}$, $w \in f_i(s)$ iff $\langle \sigma_s(v_{i_1}), \dots, \sigma_s(v_{i_n})\rangle \in I(w, F)$, for any set of variables, $v_{i_1}, \dots, v_{i_n}$.

A **general first-order neighborhood model with constant domain** is a structure $\mathcal{M}^g = \langle W, N, D, A, \{f_i\}_{i\in\omega}, I\rangle$, where $\langle W, N, D, A, \{f_i\}_{i\in\omega}\rangle$ is a general first-order neighborhood frame with constant domain and I is an A-admissible interpretation for $\{f_i\}_{i\in\omega}$. Truth and validity are defined as usual. It is easy to see that the following is true in any such model:

> $(*)$ for any scheme ϕ with n-free variables $v_{i_1}, \dots, v_{i_n}$, there is a function f_i that corresponds to the scheme $\phi(v_1, \dots, v_n)$ in the sense that for each assignment σ, $(\phi(v_1 \dots, v_n))^{\mathcal{M},\sigma} = f(s_\sigma)$.

Lemma 6.2 *For each formula $\phi \in \mathcal{L}_1$ and any general first-order neighborhood model with constant domain $\mathcal{M}^g = \langle W, N, D, A, \{f_i\}_{i\in\omega}, I\rangle$, $(\phi)^{\mathcal{M},\sigma} \in A$ for all assignments σ.*

Theorem 6.3 *Let Λ be any classical first-order modal logic. Λ is sound and strongly complete with respect to the class of general first-order neighborhood frames with constant domains for Λ.*

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A Classical Propositional Modal Logic

In this section we give a brief overview of classical propositional modal logic. The reader is referred to the textbook [11] for a complete discussion.

Let Φ_0 be a countable set of propositional variables. Let $\mathcal{L}(\Phi_0)$, be the standard propositional modal language. That is $\phi \in \mathcal{L}(\Phi_0)$ iff ϕ has one of the following syntactic form,

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi$$

where $p \in \Phi_0$. Use the standard definitions for the propositional connectives $\vee, \rightarrow$ and $\leftrightarrow$ and the modal operator $\Diamond$. The standard propositional language may be denoted $\mathcal{L}$ when Φ_0 is understood.

Definition A.1 *A* **neighborhood frame** *is a pair $\langle W, N\rangle$, where W is a set of states, or worlds, and $N : W \rightarrow 2^{2^W}$ is a function.*

Given a neighborhood frame, $\mathcal{F} = \langle W, N\rangle$, the function N is called a **neighborhood function**.

Definition A.2 *Given a neighborhood frame $\mathcal{F} = \langle W, N\rangle$, a model based on $\mathcal{F}$ is a tuple $\langle \mathcal{F}, V\rangle$, where $V : \Phi_0 \rightarrow 2^W$ is a valuation function.*

Given a model $\mathcal{M} = \langle W, N, V\rangle$, truth is defined as follows, let $w \in W$ be any state:

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$ where $p \in \Phi_0$
2. $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
3. $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Box\phi$ iff $(\phi)^{\mathcal{M}} \in N(w)$

where $(\phi)^{\mathcal{M}} \subseteq W$ is the set of all states in which ϕ is true. The dual of the modal operator $\Box$, denoted $\Diamond$, will be treated as a primitive symbol. The definition of truth for $\Diamond$ is

$$\mathcal{M}, w \models \Diamond\phi \text{ iff } W - (\phi)^{\mathcal{M}} \notin N(w)$$

It is easy to see that given this definition of truth, the axiom scheme $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is valid in any neighborhood frame. Thus, in the presence of the E axiom scheme (see below) and a rule allowing substitution of equivalent formulas (which can be proven using the RE rule given below), we can treat $\Diamond$ as a defined symbol. As a consequence, in what follows we will not provide separate definitions for the $\Diamond$ operator since they can be easily derived. We say ϕ is valid in $\mathcal{M}$ iff $\mathcal{M}, w \models \phi$ for each $w \in W$. We say that ϕ is valid in a neighborhood frame $\mathcal{F}$ iff ϕ is valid in all models based on$\mathcal{F}$.

The following axiom schemes and rules have been widely discussed.

PC Any axiomatization of propositional calculus

E $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

M $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

N $\Box\top$

RE $\dfrac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$

MP $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$

E is the smallest set of formulas closed under instances of PC, E and the rules RE and MP. The logic **EC** extends **E** by adding the axiom scheme C. Similarly for **EM**, **EN**, **ECM**, and **EMCN**. It is well known that the logic **EMCN** is equivalent to the normal modal logic **K** (see [11] page 237). Let **S** be any of the above logics, we write $\vdash_{\mathbf{S}} \phi$ if $\phi \in \mathbf{S}$.

Let N be a neighborhood function, $w \in W$ be an arbitrary state, and $X, Y \subseteq W$ be arbitrary subsets.

(m) If $X \cap Y \in N(w)$, then $X \in N(w)$ and $Y \in N(w)$

(c) If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$

(n) $W \in N(w)$

It is easy to show (see [11] page. 215) that (m) is equivalent to

(m') If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$

We say that a neighborhood function N is **supplemented**, **closed under intersection**, or **contains the unit** if it satisfies (m) (equivalently if it satisfies (m')), (c) and (n) respectively.

Definition A.3 *A frame $\langle W, N\rangle$ is* **augmented** *if N is supplemented and for each $w \in W$,*

$$\bigcap N(w) \in N(w)$$

Call a frame supplemented if its neighborhood function is supplemented, similarly for the other semantic properties above. It is well-known that the logic **E** is sound and complete with respect to the class of all neighborhood frames. The other semantic conditions correspond to the obvious syntactic counterparts. For example, the logic **EMC** is sound and complete with respect to the class of all frames that are supplemented and closed under intersection. The completeness proofs are straightforward and are discussed in [11]. One final note about the propositional case will be important for this paper. The class of augmented frames is equivalent to the class of Kripke frames in the following sense.

Theorem A.4 ([11] page 221) *For every Kripke model $\langle W, R, V\rangle$, there is an pointwise equivalent classical model $\langle W, N, V\rangle$, and vice versa*