Decisions Under Subjective Information

Jack Stecher*

Abstract

This paper addresses two closely related aspects of subjective information. First, no two agents necessarily see the same thing when they observe the same object. Second, no two agents have the same concepts in mind when they use the same term in a shared language. I present a framework to give rigorous meaning to these ideas, in particular by modeling them in appropriate topologies. From there I give conditions under which a shared language enables agents with subjective information to trade. These conditions turn out to be equivalent to a form of continuity of the mappings between agents' perceptions and their use of the shared language, when both are viewed as topological spaces.

1 Introduction

For economic activity to occur, both parties in a trade need to have a shared understanding of what is being exchanged. The usual approach in economic theory is to assume the existence of a commodity space, the nature of which is common knowledge. Agents might observe elements of the commodity space with some random noise, and in some models might not be aware of the entire commodity space (Dekel, Lipman and Rustichini 1998a, Dekel, Lipman and Rustichini 1998b, Ruffle and Kaplan 2000). However, if different agents see the same thing, they are assumed to agree on what they observe. Moreover, agents typically have their preferences defined on the commodity space; this is natural only because there is no distinction between a commodity basket's objective characteristics and an agent's subjective perception of the commodity basket.

In many settings, however, an agent may perceive the same object in many ways. For example, suppose two bottles of wine differ by trace amounts in their sugar and glycerin concentrations. For small enough differences, the bodies of the two wines will be indistinguishible even to a wine connoisseur. Psychologists describe this phenomenon by saying that the wines are within a just-noticeable difference (jnd). In this example, if the heavier wine tastes full-bodied to the connoisseur, then the lighter one, by virtue of being within a jnd for the connoisseur of the heavier one, is also perceptible as full-bodied.

Imagine a third bottle of wine, lighter than either of the two described above. The body of the third bottle may differ by more than a jnd from the heaviest wine, yet be within a jnd of the second-heaviest. If the third bottle tastes medium-bodied to the connoisseur, then the second-heaviest wine, by virtue of being indistinguishible from the lightest wine, must also be perceptible as medium-bodied. Thus, the connoisseur may perceive the same bottle of wine as either full-bodied or medium-bodied. Putting that another way, the connoisseur's subjective experience of the wine is not uniquely determined by the wine's objective characteristics.

*Carlson School of Management—University of Minnesota, Department of Accounting, Room 3-122, 321–19 Avenue South, Minneapolis, MN 55455. Email: jstecher@csom.umn.edu.

A further complication arises when the connoisseur decides whether to try a new wine, and asks a novice wine steward whether the new wine is full-bodied. The novice, whether on account of inexperience or natural ability, has a larger jnd than the connoisseur. Accordingly, there will be many wines that the connoisseur could distinguish from a full-bodied wine, but which would taste identical to the novice.

Since neither agent has access to the other's perceptions, any exchange of information must occur in a shared language, which can be thought of as a standardized reporting system (such as a set of financial accounting rules, conventions for listing items on menus, etc.). The vocabulary of the language and the syntax may be common knowledge; however, each agent will interpret the vocabulary subjectively. Thus, the connoisseur will have some ideas about which wines can be called full-bodied, if there is such a term in the language. After a few unpleasant surprises from trading with novice wine stewards, the connoisseur is likely to infer that the term *full-bodied* is commonly used only to make coarse distinctions.

There are thus two related forms of subjectivity addressed here. First, there is an agent's *perception* of a real-world object. This is modeled as a binary relation between real-world objects and the agent's subjective conceptions of the world. Two objects are thought of as being within a jnd if and only if they can be perceived the same way, i.e., iff they both belong to the relation for the same conception.

Second, there is an agent's method of *reporting* an object proposed in trade. Reporting is modeled as a binary relation between the agent's subjective conceptions of the world and the terms in the shared language. The accountant's notion of an immaterial difference plays a role in reporting analogous to that of a jnd in perception: two subjectively experienced objects differ immaterially if and only if they can be reported in the same way.

Fundamental to both these relations is the agent's set of subjective conceptions; this is also the set on which the agent's preferences are defined. That is, preferences are defined on the world as the agent subjectively experiences it, and not on how commodities are described or how they appear to an omniscient observer. Thus, the physical commodity space, the set where preferences are defined, and the set where trade occurs are all different entities.

The remainder of this paper makes the above ideas precise and describes the results that can be obtained. Of particular interest is the implication for how precise the shared language should be (e.g., what sort of distinctions should a standard-setting body enable an agent to make in a reporting system). Too fine-grained a language leads to coordination failures: in the above example, it is easy to see that if the connoisseur can describe a wine's body in greater detail than the steward can distinguish, the language will not prevent agents from getting something in trade that differs from what they thought they bargained for. On the other hand, if the language is made overly coarse, coordination failures will arise from the inability of agents to request what they want.

Existing research has found mixed results on imprecision in language. Lipman (2001) shows that, under standard neoclassical assumptions, vague language cannot be desirable. Consequently, Lipman views vague language as explained by departures from neoclassical theory such as bounded rationality. On the other hand, Allen and Thisse (1992) find that pure strategy Nash equilibria are possible when agents do not care about small price deviations; that is, imprecision in prices takes the place of mixing.

The chief reason for the different results here from those in the literature is that each agent needs to interpret what is traded in the share language into the agent's private subjective world, and not into its objective meaning in an agreed-upon commodity space. Agents can disagree on what is being traded, and it is this feature that makes imprecise language desirable.

2 Perceptions

Assume there is a set I of agents. For each $i \in I$, there is an associated set S_i, called i's *conceptions*, with the intended interpretation as the agent's understanding of what may potentially be in the world. The collection of potential alternatives is written X, interpreted as real-world objects as they would appear to an objective, omniscient observer. No agent observes X, and for $i \neq j$, agent i cannot observe S_j.

Each agent has a binary relation $\Vdash_i$ between X and S_i.[1] For $x \in X$ and $a \in S_i$, read $x \Vdash_i a$ as "i can perceive x as a." Assume that every object can be perceived in some way—that is, the agent has a coarse enough conception to cover every object:

$$(\forall x \in X)(\exists a \in S_i) \text{ such that } x \Vdash_i a.$$

Additionally, assume that, for every $a, b \in S_i$, if there is something perceptible both as a and as b, then the agent can conceptualize the overlap of these two conceptions:

$$(\forall x \in X)(\forall a, b \in S_i)((x \Vdash_i a \text{ and } x \Vdash_i b) \text{ implies}$$

$$((\exists c \in S_i)(\forall y \in X)(y \Vdash_i c \text{ implies } y \Vdash_i a \text{ and } y \Vdash_i b))).$$

This structure describes how an agent can perceive each individual object. Additionally, an agent may have a perception of a set of real-world objects. For $D \subseteq X$, a weak sense of the way agent i perceives D is the set of ways some object in D could be perceived, written $\Diamond_i D$:

$$\Diamond_i D \equiv \{a \in S_i | (\exists x \in X)(x \Vdash_i a \text{ and } x \in D)\}.$$

A stronger view of i's perception of D is as the set of perceived objects that can only be observed when viewing a real-world object in D, written $\Box_i D$:

$$\Box_i D \equiv \{a \in S_i | (\forall x \in X)(x \Vdash_i a \text{ implies } x \in D)\}.$$

An example of where the strong image $\Box_i D$ seems appropriate is when D is a constraint set, since $\Box_i D$ represents the perceived choices that necessarily satisfy the constraints. On the other hand, a consumer choosing some object in D might perceive the choice as something in $\Diamond_i D$ but not necessarily in $\Box_i D$. In this case, the weak image of D seems more appropriate.

Both $\Diamond_i$ and $\Box_i$ give the agent's subjective view of a set of real-world objects. Suppose instead that agent i has in mind a set $U \subseteq S_i$ of perceived objects. An omniscient[2] researcher studying this agent may be interested in the set of real-world objects that the agent might perceive as something in U; this set is called the *extent* of U:

$$\text{ext}_i\, U \equiv \{x \in X | (\exists a \in S_i)(x \Vdash_i a \text{ and } a \in U)\}.$$

The set of real-world objects necessarily perceived as something in U is called the *restriction* of U, defined as follows:

$$\text{rest}_i\, U \equiv \{x \in X | (\forall a \in S)(x \Vdash_i a \text{ implies } a \in U)\}.$$

Observe that ext_i is defined exactly symmetrically to $\Diamond_i$. Each gives the weak (existential) image of a set along the agent's perceptual relation $\Vdash_i$. Similarly, rest_i and $\Box_i$ are symmetric, giving the strong (universal) image of a set along $\Vdash_i$.

[1] This is a forcing relation; see Avigard (2004) or Moore (1988) for general background and Sambin (2001) for details in a context related to the present one.

[2] In the sense of knowing all of X.

Consider in particular the extent of $\Box_i D$ for some collection $D \subseteq X$ of real-world objects. Since $\Box_i D \subseteq S_i$ is a collection of agent i's perceived objects, the definition of extent gives:

$$\text{ext}_i\ \Box_i D = \{x \in X | (\exists a \in S_i)(x \Vdash_i a \text{ and } a \in \Box_i D)\}.$$

Expanding the definition of $\Box_i$ then yields:

$$\text{ext}_i\ \Box_i D = \{x \in X | (\exists a \in S_i)(x \Vdash_i a \text{ and } (\forall y \in X)(y \Vdash_i a \text{ implies } y \in D))\}.$$

This set has a natural interpretation. Intuitively, two real-world objects are "close" in terms of the agent's perceptions if the agent can perceive them the same way. That is, $x, y \in X$ are perceptually close for agent i if, for some $a \in S_i$, both x and y are perceptible as a. Thus, the set $\text{ext}_i\{a\}$ represents all of the points in X that are in a sense close to each other: mathematically, these are the points in some neighborhood, and a is the agent's representation of this neighborhood. That is, the set S_i are agent i's representation of the basis of a topology on X.[3]

Bearing this interpretation in mind, the definition of $\text{ext}_i\ \Box_i D$ is read as follows: the set of all points in X which belong to an open neighborhood (associated with $a \in S_i$) which is entirely contained in D. That is,

Definition 2.1 (Perceptual Topology)**.** *The* perceptual interior *of $D \subseteq X$ is*

$$\text{int}_i\ D \equiv\ \text{ext}_i\ \Box_i D.$$

A subset D of X is open *iff $D =\ \text{int}_i\ D$. The collection of open sets under this definition is called agent i's* perceptual topology.

It is then straightforward to show that this definition is sensible:

Theorem 2.1. *The perceptual topology is indeed a topology. That is, it is closed under arbitrary union and under finite intersection, and both X and $\varnothing$ are open.*

Closure is defined dually to interior: the closure of $D \subseteq X$ is defined as

$$\text{cl}_i\ D \equiv\ \text{rest}_i\ \Diamond_i D.$$

This says that the closure of $D \subseteq X$ is the set of points in X which can exclusively be perceived in ways that are possibly the perception of something in D. Topologically speaking, this says that $x \in\ \text{cl}_i\ D$ iff every open neighborhood of x intersects D.[4]

Remark: In general, $\text{int}_i\ D \bigcup \text{cl}_i\ D^c \neq X$. That is, the union of a set's interior with the closure of its complement does not exhaustively describe the entire space. Thus the perceptual topology is in general an intuitionistic topology. The intuition behind this result is that the agent cannot perceive boundaries clearly.

3 Reporting

Now consider what happens when agents meet to discuss potential trades. Unless the physical commodities are present at the meeting, neither agent can access members of the commodity space

[3]This interpretation is similar to that in Valentini (2001), Sambin (2003), and Maietti and Valentini (2004) in a different context.

[4]This is related to the concept of nearness in Vîţă and Bridges (2003).

X directly. Moreover, neither agent can peek inside the other agent's head; that is, each agent's conceptions are private.

Assume, however, that the agents use a language with a shared set of terms T. Each agent $i \in I$ has a binary relation R_i, interpreted as reporting or entailment, between S_i and T. Thus, if $a \in S_i$, $t \in T$, and aR_it, the the intended meaning is "agent i can report private conception a using term t." Assume that there is some $t \in T$ with aR_it for every $a \in S_i$; this means that the agent has some valid way of reporting any conception, though of course these terms need not apply only to the conception that the agent has in mind. Similarly, assume that if there are two terms t, t' such that a conception a is reportable as both t and t', then the agent can express the idea of something that satisfies the definitions of both terms. I.e., the agent can express some $t'' \in T$ such that everything reportable as t'' is reportable as both t and t'.

The relationship between the agent's conceptions and the shared language is easily seen to be analogous to that between the real world and the agent's conceptions: there are two collections, a binary relation between them, and the same minimal assumptions on the relation. Therefore, the following result can be shown with arguments that are completely symmetric to those in the previous section:

Theorem 3.1 (Reporting Topology)**.** *Each agent's reporting relation R_i induces an intuitionistic topology on the agent's conceptions S_i. The set T of shared terminology can then be identified with the names of the members of a basis for this topology.*

Intuitively, two conceptions differ immaterially if they can be reported in the same way; mathematically, this says that conceptions are close in the reporting topology iff they are in the same neighborhood in the basis. Thus, differing materially is analogous to differing by more than a jnd.

When two agents meet, they can trade with each other using the shared language only if what is promised in T corresponds to what each agent believes subjectively is under discussion. Because neither the perception nor the report of any given object is unique, there is no way to guarantee that what one agent promises is exactly what another anticipates. The best that can be hoped for, then, is that what one agent promises is something the other agent can justify. That is, given what an agent is promised, what was actually received in trade must be something the agent can perceive as reportable as advertised.

This notion of the ability to trade then says that agents i and j can trade $x \in X$ iff, for any $t \in T$, x is reportable by i as t iff it is reportable by j as t. The requirement for i and j to be able to trade arbitrary objects is then that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Vdash_i} & S_i \\ {\scriptstyle \Vdash_j}\big\downarrow & & \big\downarrow{\scriptstyle R_i} \\ S_j & \xrightarrow{R_j} & T \end{array}$$

I.e., agents i and j can trade using T iff $R_i \circ \Vdash_i = R_j \circ \Vdash_j$, where the relations are interpreted here as their associated correspondences.

Since the correspondences above are not in general functions, they always have inverses—indeed, these inverses were implicitly used above in the definitions of $\Diamond_i$ and $\Box_i$. It is straightforward to show that the above diagram commutes iff the inverse diagram commutes—i.e., if the diagram obtained by reversing all arrows and replacing the correspondences with their inverses commutes (so that $\Vdash_i^{-1} \circ R_i^{-1} = \Vdash_j^{-1} \circ R_j^{-1}$). This observation leads to the following:

Theorem 3.2. *Two agents i and j can trade using T iff agent j's perceptions $\Vdash_j$, viewed as a correspondence between topological spaces $\langle X, S_i \rangle$ and $\langle S_j, T \rangle$, is lower hemicontinuous.*

In other words, the inverse of the map from X to S_j induced by agent j's carries open neighborhoods of S_j in the reporting topology to open neighborhoods of X in agent i's perceptual topology.

The ability of agents to trade is thus a form of continuity, and the topological interpretation makes this clear. The result is sensible, in that total mappings in intuitionistic topological spaces are naturally associated with continuity. It can also be interpreted as a stability condition in how agents report, in that lower hemicontiuity is closely related to strategic stability (Kohlberg and Mertens 1986).

4 Preferences and Trade

The continuity condition stated above is necessary for agents with subjective perceptions and subjective semantics to trade using the shared language T. To discuss whether agents are willing to trade, and not simply capable of trading, requires a discussion of preferences.

Unlike in neoclassical consumer theory, however, preferences are not defined on either the shared language T or on the objective alternatives in X. Because agents experience the world subjectively, the most appropriate place to define their preference relations is on the set of subjective conceptions S_i.

In general, preferences are best modeled in this setting as incomplete. There are several reasons for this: first, there are the assumptions on perceptions, which in particular state that some conceptions may overlap, and when they do, there is a conception of (a subset of) their overlap. This immediately leads to problems with completeness; for a discussion, see Stecher (2005). Intuitively, if c is a special case of both a and of b, it will be difficult to assert that a can be strictly preferred to b. On the other hand, if S_i is connected in an appropriate topological sense, then complete preferences would force indifference to be intransitive (unless agent i is indifferent to everything). A second, related reason for expecting preferences to be incomplete is that the appropriate form of logic for reasoning about S_i seems to be intuitionistic.

Nevertheless, it is possible to define a reflexive and transitive relation $\not\succ_i$ on S_i, and an irreflexive and symmetric relation $\nsim_i$, such that the two relations form a virtual preorder on S_i. That is,

$$a \not\succ_i b \text{ implies } \neg(a \nsim b \text{ and } b \not\succ a),$$

and

$$c \not\succ_i b, b \not\succ_i a, \text{ and } b \nsim_i a \text{ imply } c \nsim_i a;$$

$$c \not\succ_i b, c \nsim_i b, \text{ and } b \not\succ_i a \text{ imply } c \nsim_i a.$$

It can be shown that the assumptions on $\Vdash_i$ and on R_i also hold, appropriately translated, when the binary relation is $\not\succ_i$. Thus, the following holds:

Theorem 4.1. *Agent i's preferences induce an intuitionistic topology on S_i.*

Note that this topology is closely connected with the Alexandroff topology, i.e., that where upper contour sets are defined as open sets.[5] A topological representation of incomplete preferences is in general the best that can be hoped for; for the failure of a utility representation, see Fishburn (1976), Dubra, Maccheroni and Ok (2004), and Stecher (2005).

[5] See Troelstra and van Dalen (1988), Vickers (1988), or Bridges and Mehta (1995).

The desirability of trade is now essentially a repitition of the continuity argument in the preceding section. An agent is willing to trade for something in S_i that is better than the perceived endowment. An offer $t \in T$ must therefore have an interpretation in S_i as something in the upper contour set of i's endowment. Thus, continuity describes the willingness and the ability of an agent to engage in trade.

5 Concluding Remarks

This paper models agents who perceive the world subjectively, whose preferences are defined on their subjective conceptions of their alternatives, and who interpret reports in a shared language subjectively. A rich structure emerges which enables one to discuss conditions on a shared language that enable agents to trade. In particular, perceptions, preferences, and reporting are all associated with topological structures, and the ability for agents to engage in trade then becomes a form of continuity.

Despite the positive result that states how trade can occur, there remains the possibility that trade will be welfare-reducing. There are several reasons for this, but they are all related to the idea that agents do not know exactly what they are getting when they trade; the best they can hope to obtain is approximate information.

This leads to a natural question for follow-up research: how precise should a standard-setting body (e.g., one designing a financial accounting system or similar set of reporting rules) try to make a reporting language? It would seem to be catastrophic if the language were overly hair-splitting, as such a reporting system would lead agents to receive in trade items that are unlike what they imagined possible. (Think about the esoteric distinctions that Enron's executives used when justifying their reports, and ask whether these led Enron to use terms like *revenue* in ways that matched investors' understanding of these words.) The possibility that an imprecise reporting language could be welfare-improving has been explored in the accounting literature (Kanodia, Singh and Spero 2004), but in a very different context. The arguments contained here may shed light on this issue.

At the same time, it is clear that enabling agents to use the same terms to mean the same thing cannot be a sufficient criterion for any sort of problem in the optimal design of a reporting system. If it were, there would never be any reason to move beyond a universal language, i.e., one with a single term such as *something* or *stuff*. Thus there is a trade-off between potential coordination catastrophes and terminology too vague to be useful.

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